The Structure and Visualization of Performance Attribution

The author proposes another alternative to current multi-period performance attribution methods. In addition to the text, the author has included considerable data in tables to support his thesis.

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This paper presents a simply additive, yet formally exact, approach to multi-period performance attribution. It also introduces a visual representation, which makes this new approach and its relationship to prior approaches intuitively clear.

Most approaches to performance attribution only apply to an average over a single time period (Brinson, 1985, 1986, and 1991; Karnosky, 1994; and Frank Russell Company, 1994) or incorporate compounding over time periods by building approximations or residuals into their foundational premises (Karnosky, 1995; Singer, 1996, and 1998; Los, and 1999; Kirievsky, 2000).[1] Some other prominent multi-period approaches are exact but unintuitive or incomplete (Burnie, 1998). A recently published approach (Carino, 1999) escapes or ameliorates these deficiencies, but it incorporates a new major drawback: It makes the evaluation of any month's contributions to this year's performance depend upon the values of properties that are not available until next year. For example, August's stock selection contribution to this year's excess return does not become available until after the end of December. This article presents a new approach to performance attribution, which avoids all the problems mentioned, and is easily understood by means of simple diagrams.

THE CHALLENGE OF PERFORMANCE ATTRIBUTION

Performance attribution is the general name given to a variety of approaches aimed at understanding how a portfolio achieved its results. All the approaches to performance attribution discussed below analyze the excess return of a portfolio relative to its benchmark, over a given time period, by decomposing the excess return into elements. Each of these elements is meant to measure the contribution to this excess return attributable to a particular type of investment decision.[2] For example, the (stock) selection element aims to measure the contribution to the excess return caused by the decision to weight stocks within their segment of the investment universe differently than does the benchmark. The allocation element aims to measure the contribution due to the decision to weight the segments themselves differently than does the benchmark. A complete set of contributions comprises the portfolio's total excess return for the period.

It is a challenging task to construct a satisfactory analysis of the excess return into meaningful elements. Temporal compounding (linking returns over time) can misleadingly introduce, complicated dependencies. Whereas, for the decomposition of excess return to be useful, it must result in a simple set of intuitively meaningful and appropriately available elements. Only an exceedingly accessible and timely approach can successfully inform portfolio managers and others who need to quickly understand the reasons for the recent achievements of a portfolio. In short, a successful theory of performance attribution should have no built-in errors due to inexact compounding;[3] the elements should simply add, not multiply, together to give the correct excess compounded return, in order to produce results that are easily informative; and, to be timely, the ultimate value of the constituents of a sub-period within a whole period should be calculable at the end of the sub-period, instead of requiring information that becomes available only much later, at the end of the whole period.

For example, in September portfolio managers want to know, for the preceding month of August, the ultimate and exact industry allocation, selection and other contributions to the excess return of a portfolio for the total calendar year. They cannot wait till next year to find out the values of this August's exact contribution to this year's excess return. Furthermore, all of these contributions, without residual, should exactly sum to the total excess return. Analogous relationships should hold at both the issue and sector levels.

Previous exact approaches (Carino, 1999) to multi-period performance attribution decompose the excess geometric (i.e. compounded) return of a portfolio relative to its benchmark into constituents constructed out of components defined arithmetically (i.e. without compounding). This mismatch, between the geometric form of the total and the arithmetic form of the parts, leads to exact decompositions that are inherently complicated and ad hoc. It turns out to be considerably simpler, and to lead to more intuitive results, to decompose the excess geometric return into constituents that are themselves geometrically derived. Employing this insight, this paper presents a new mirroring approach to performance attribution, which is simple, timely and exact. The paper goes on to demonstrate how a number of approaches to performance attribution, including the favored new mirroring one introduced here, can be visualized and understood by means of intuitively accessible diagrams.

Standard Preliminaries

Choosing the length of the basic sub-period to be a day, the geometric return, ${}^{g}R_{1\text{ to }T}$, of the portfolio for the period, including all of day 1 through all of day $T \geq 1$, is obtained in terms of the portfolio's return, R_t, for each individual day (t), by means of a product:

$$(1 + {}^{g}R_{1\text{ to }T}) \equiv (1 + R_1)*(1 + R_2)*(1 + R_3)*\ldots$$

$$*(1 + R_{T-1})*(1 + R_T) = \Pi_{t=1\text{ to }T}(1+R_t).$$

The geometric factor, $(1 + {}^{g}R_{1\text{ to }T})$, for the period, T, is incremented on the succeeding day, T+1, by the incremental geometric factor, $(1 + R_{T+1})$, for that day:

$$(1 + {}^{g}R_{1\text{ to }T+1}) = (1 + {}^{g}R_{1\text{ to }T}) * (1 + R_{T+1}).$$

Herein, "incremental" will always refer to a change in a value over one single day.

Denoting benchmark values by a superscript suffix 'B', the benchmark's geometric return for the period, from the start of day 1 to the end of day T, is represented by ${}^{g}R^{B}_{1\text{ to }T}$.

As performance attribution is herein understood, the excess geometric return, which it is the purpose of performance attribution to analyze, is the amount that the geometric return, ${}^{g}R_{1\text{ to }T}$, of the portfolio for the period is in excess over the geometric return, ${}^{g}R^{B}_{1\text{ to }T}$, of the benchmark for the same period. This "excess" can be interpreted either geometrically as $(1 + {}^{g}R_{1\text{ to }T})/(1 + {}^{g}R^{B}_{1\text{ to }T}) - 1 = ({}^{g}R_{1\text{ to }T} - {}^{g}R^{B}_{1\text{ to }T})/(1 + {}^{g}R^{B}_{1\text{ to }T})$, giving the geometric excess of geometric returns, or arithmetically as ${}^{g}R_{1\text{ to }T} - {}^{g}R^{B}_{1\text{ to }T}$, giving the arithmetic excess of geometric returns. In either interpretation, "excess" always refers to the comparison of the portfolio to the benchmark.

THE TWO-STEP DECOMPOSITION OF EXCESS RETURN

The basic decomposition of the excess return often can be thought of as being carried out in two steps. First, the excess return for the period is obtained as a concatenation of constituents defined in terms of incremental, single-day, returns. Subsequently, these constituents are themselves obtained in terms of contributions due to decision types. (As an aside, the contribution for a single decision type can be accumulated over days in the period.) This two-step process results in a decomposition of the period's excess return into a set of elements involving the daily contributions due to decision types.

In order to carry out the first step of this decomposition, most previous standard exact approaches to performance attribution decompose the excess geometric return for the period into a function of the excess arithmetic incremental (single-day) returns.[4] The arithmetic return for a period, from the start of day 1 to the end of day t, is the sum of the single-day returns for the days in the period: ${}^{a}R_{1\ to\ t} = \Sigma_{\tau = 1\ to\ t}\,(R_\tau)$. The incremental arithmetic return is the amount that the arithmetic return for the period changes in one day, which is trivially seen to be the return for the single day, R_t:

$${}^{a}R_{1\ to\ t} - {}^{a}R_{1\ to\ t-1} = [\Sigma_{\tau = 1\ to\ t}\,(R_\tau)] - [\Sigma_{\tau = 1\ to\ t-1}\,(R_\tau)] = R_t.$$

The arithmetic excess of the incremental arithmetic returns is the difference between the incremental arithmetic return for the portfolio and for the benchmark:

$${}^{a}\Delta^{a}_{t} \equiv ({}^{a}R_{1\ to\ t} - {}^{a}R_{1\ to\ t-1}) - ({}^{a}R^{B}_{1\ to\ t} - {}^{a}R^{B}_{1\ to\ t-1}) =$$

$$\{[\Sigma_{\tau = 1\ to\ t}\,(R_\tau)] - [\Sigma_{\tau = 1\ to\ t-1}\,(R_\tau)]\} - \{[\Sigma_{\tau = 1\ to\ t}\,(R^{B}_{\tau})] -$$

$$[\Sigma_{\tau = 1\ to\ t-1}\,(R^{B}_{\tau})]\} = R_t - R^{B}_{t}.$$

It is this term, ${}^{a}\Delta^{a}_{t}$, the arithmetic excess of the incremental arithmetic returns, which these previous standard exact approaches chose to make the foundation for their decomposition of the excess geometric return.

After obtaining the excess geometric return for the period as a concatenation of daily constituents that explicitly incorporate, ${}^{a}\Delta^{a}_{t}$, the arithmetic excess of the incremental arithmetic returns for each day in the period, the previous standard approaches carry out the second step of this decomposition by decomposing this arithmetic excess of the incremental arithmetic returns for the day into a sum of contributions, each associated with an individual decision type:

$${}^{a}\Delta^{a}_{t} = A_t + S_t + I_t.$$

The value of a contribution term is intended to represent the extent of the impact, for day t, of the particular type of financial decision on the excess return of the portfolio. For example, A_t is the contribution to the excess arithmetic return of day t associated with all the allocation decisions on that day.

Next, these previous standard approaches combine the two steps. The first step was building the excess geometric return for the period, $(1 + {}^{g}R_{1\ to\ T})/(1 + {}^{g}R^{B}_{1\ to\ T}) - 1$, or ${}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T}$, out of constituents which directly incorporate the arithmetic excess of the incremental arithmetic returns, ${}^{a}\Delta^{a}_{t}$. The second step was building the arithmetic excess of the incremental arithmetic returns, ${}^{a}\Delta^{a}_{t}$, out of the contributions, A_t, S_t, and I_t, associated with decision types. Combining these, they obtain the geometric excess return in terms of decision-specific constituents, which are functions of these contributions.

The First Step: The Decomposition of Excess Return in Terms of an Excess Incremental Return

Three Previous Approaches

In order to introduce, motivate and lay the ground work for explaining and evaluating the successful new mirroring approach to decomposing excess return into components incorporating excess incremental returns, the structure of three previous approaches is analyzed.

AAAA:

- The arithmetic excess of arithmetic returns as a simple direct sum of the arithmetic excess of the incremental arithmetic returns;

$${}^{a}R_{1\ to\ T} - {}^{a}R^{B}_{1\ to\ T} = \Sigma_{t = 1\ to\ T}\,(R_t - R^{B}_{t}) = \Sigma_{t = 1\ to\ T}\,\{{}^{a}\Delta^{a}_{t}\},$$

$$\text{where, } {}^{a}\Delta^{a}_{t} \equiv ({}^{a}R_{1\ to\ t} - {}^{a}R_{1\ to\ t-1}) - ({}^{a}R^{B}_{1\ to\ t} - {}^{a}R^{B}_{1\ to\ t-1}),$$

$$\text{with } {}^{a}R_{1\ to\ t} \equiv \Sigma_{\tau = 1\ to\ t}\,(R_\tau).$$

As a preliminary case, consider the decomposition of the arithmetic excess of the arithmetic returns in terms of the arithmetic excess of the incremental arithmetic returns. It follows directly from the definition of the arithmetic return, ${}^{a}R_{1\ to\ T}$, for period T that:

$${}^{a}R_{1\ to\ T} - {}^{a}R^{B}_{1\ to\ T} = \Sigma_{t = 1\ to\ T}\,\{\,R_t - R^{B}_{t}\,\}.$$

That is, the arithmetic excess of the arithmetic returns for the period is just the sum, over days in the period, of the arithmetic excess of the daily returns for each of the days in the period.

Employing the equation that relates the arithmetic excess of the incremental arithmetic returns to the arithmetic excess of the daily returns, ${}^a\Delta^a_t = R_t - R^B_t$, the arithmetic excess of arithmetic returns for the period simply becomes the sum over time of the arithmetic excess of the incremental arithmetic returns:

$$^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\{{}^a\Delta^a_t\}.$$

This historically original approach decomposes the arithmetic excess of the arithmetic returns, ${}^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T}$, into the direct simple sum of constituents, in {} brackets. In this case, each such constituent, {}, is ${}^a\Delta^a_t$, the arithmetic excess of the incremental arithmetic returns for the period on day t. These are the components that will later be analyzed into contributions, which are individually associated with a particular decision type on a particular day.

The goal of exact performance attribution is to decompose the excess geometric return, ${}^gR_{1\text{ to }T} - {}^gR^B_{1\text{ to }T}$, not to decompose its arithmetic approximation, ${}^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T}$. Since this approach (**AAAA**) ignores compounding by only decomposing the excess arithmetic return, ${}^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T}$, it is not exact. It is noted that components, ${}^a\Delta^a_t$, are inherently arithmetic in that by themselves they directly compose this arithmetic approximation, ${}^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T}$.

GGAA:

- The geometric excess of geometric returns plus one as a product of a function of the arithmetic excess of the incremental arithmetic returns;

$$(1 + {}^gR_{1\text{ to }T})/(1+ {}^gR^B_{1\text{ to }T}) = \Pi_{t=1\text{ to }T}\{\exp[\ln(1 + R_t) - \ln(1+ R^B_t)]\} = \Pi_{t=1\text{ to }T}\{\exp(k_t * {}^a\Delta^a_t)\},$$

$$\text{where, } {}^a\Delta^a_t \equiv ({}^aR_{1\text{ to }t} - {}^aR_{1\text{ to }t-1}) - ({}^aR^B_{1\text{ to }t} - {}^aR^B_{1\text{ to }t-1}),$$

$$\text{with } {}^aR_{1\text{ to }t} \equiv \Sigma_{\tau=1\text{ to }t}(R_\tau).$$

The decomposition of the geometric excess of the geometric returns in terms of the arithmetic excess of the incremental arithmetic returns has as an initial move the following decomposition:

$$(1 + {}^gR_{1\text{ to }T})/(1 + {}^gR^B_{1\text{ to }T}) =$$

$$\Pi_{t=1\text{ to }T}\{\exp[\ln(1 + R_t) - \ln(1+ R^B_t)]\},$$

i.e., one plus the geometric excess of the geometric returns for the period is the product of terms, one for each day, t, in the period, of the exponential, (or anti-log) of the arithmetic excess of the logarithms of one plus the daily return.

Again employing the equation which relates the arithmetic excess of the incremental arithmetic returns to the arithmetic excess of the daily returns, ${}^a\Delta^a_t = R_t - R^B_t$, ${}^a\Delta^a_t$ can be tautologically inserted into the decomposition stated above to obtain:

$$(1 + {}^gR_{1\text{ to }T})/(1+ {}^gR^B_{1\text{ to }T}) = \Pi_{t=1\text{ to }T}\{\exp[(\ln(1 + R_t) - \ln(1 + R^B_t)) * {}^a\Delta^a_t/(R_t - R^B_t)]\}.$$

(${}^a\Delta^a_t$ could not be introduced by substitution since the term $R_t - R^B_t$ does not appear in the initial decomposition.)

If k_t is defined:

$k_t \equiv [\ln(1+ R_t) - \ln(1+ R^B_t)]/(R_t - R^B_t)$, one obtains:

$$(1 + {}^gR_{1\text{ to }T})/(1 + {}^gR^B_{1\text{ to }T}) = \Pi_{t=1\text{ to }T}\{\exp(k_t * {}^a\Delta^a_t)\}.$$

This approach decomposes one plus the geometric excess of the geometric returns, $(1 + {}^gR_{1\text{ to }T})/(1 + {}^gR^B_{1\text{ to }T})$, into a product of constituents, $\{\exp[k_t * {}^a\Delta^a_t]\}$, where each individual constituent is associated with a component, ${}^a\Delta^a_t$, of the arithmetic excess of the arithmetic returns for the whole period, ${}^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T}$.

As is argued by Dr. David Carino (1999), and supported by others (Surz, 1999), a serious drawback of this geometric decomposition is that its constituents are not additive, but rather must be multiplied together to provide the total excess return.[5] Our intuitions of how various elements combine to contribute to a total effect are much better informed by an additive, as opposed to a multiplicative, composition. For example, it would be very strange indeed to say that, in some unspecified game, because I won the first quarter by two points, the second quarter by three points, the third quarter by four points and lost the last quarter by 59/60 of a point then the final result is that I tied (i.e., I won by 0 points) as deduced from the formula: $(2 + 1)*(3 + 1)*(4 + 1)*(-59/60 + 1) - 1 = 0$. (Note: When I go ahead by "x" my opponent falls behind by x/[1 + x], which means that when I multiply my score plus one by 1 + x that, my opponent's score plus one gets divided by 1 + x.) This multiplicative mode of assessment is too unintuitive to be a viable explanation of results in either sports or in performance attribution.

AGAA:

- The arithmetic excess of geometric returns as a scaled sum of the arithmetic excess of the incremental arithmetic returns;

$$
\begin{aligned}
&{}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T} = \\
&\Sigma_{t=1 \text{ to } T}\, \{[({}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T})/(\ln(1 + {}^{g}R_{1 \text{ to } T}) - \\
&\ln(1 + {}^{g}R^{B}_{1 \text{ to } T}))] * [\ln(1 + R_t) - \ln(1 + R^{B}_t)]\} \\
&= \Sigma_{t=1 \text{ to } T}\, \{(k_t/k) * {}^{a}\Delta^{a}_t\},
\end{aligned}
$$

where, ${}^{a}\Delta^{a}_t \equiv ({}^{a}R_{1 \text{ to } t} - {}^{a}R_{1 \text{ to } t-1}) - ({}^{a}R^{B}_{1 \text{ to } t} - {}^{a}R^{B}_{1 \text{ to } t-1})$,

with ${}^{a}R_{1 \text{ to } t} \equiv \Sigma_{\tau=1 \text{ to } t}\,(R_\tau)$.

An improved approach (Carino, 1999[6]) to performance attribution is to decompose the arithmetic excess of the geometric returns in terms of the arithmetic excess of the incremental arithmetic returns. The initial move in this decomposition can be rendered as follows:

$$
\begin{aligned}
&{}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T} = \\
&\Sigma_{t=1 \text{ to } T}\, \{[({}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T})/(\ln(1+{}^{g}R_{1 \text{ to } T}) - \\
&\ln(1+{}^{g}R^{B}_{1 \text{ to } T}))] * [\ln(1+R_t) - \ln(1+R^{B}_t)]\},
\end{aligned}
$$

i.e., the arithmetic excess of the geometric returns for the period is equal to the sum of terms, one for each day, t, in the period, composed of the arithmetic excess of the geometric returns for the whole period itself, ${}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T}$, multiplied by a function of logarithmic terms. Since the introduced function, consisting of all the logarithmic terms, sums to one, the equality follows immediately.

The term $R_t - R^{B}_t$ again is not present in the decomposition. However, as in the previous approach (**GGAA**), the components, ${}^{a}\Delta^{a}_t = R_t - R^{B}_t$, associated with decision types can be tautologically inserted:

$$
\begin{aligned}
&{}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T} = \\
&\Sigma_{t=1 \text{ to } T}\, \{[({}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T})/ \\
&(\ln(1 + {}^{g}R_{1 \text{ to } T}) - \ln(1 + {}^{g}R^{B}_{1 \text{ to } T}))] * \\
&[(\ln(1 + R_t) - \ln(1 + R^{B}_t))/(R_t - R^{B}_t)] * {}^{a}\Delta^{a}_t\}.
\end{aligned}
$$

Recalling:

$$k_t = [\ln(1 + R_t) - \ln(1 + R^{B}_t)]/(R_t - R^{B}_t),$$

it follows that if k is defined:

$$k \equiv [\ \ln(1+ {}^{g}R_{1 \text{ to } T}) - \ln(1+ {}^{g}R^{B}_{1 \text{ to } T})]/\ ({}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T}),$$

one obtains:

$${}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T} = \Sigma_{t=1 \text{ to } T}\, \{(k_t/k\ ) * {}^{a}\Delta^{a}_t\}.$$

Thus, this improved approach decomposes the arithmetic excess of the geometric returns, ${}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T}$, into the sum of constituents $\{(k_t/k) * {}^{a}\Delta^{a}_t\}$, where each individual constituent is again associated with ${}^{a}\Delta^{a}_t$, the arithmetic excess of the incremental arithmetic returns for the day. The inclusion of the term to be analyzed, ${}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T}$, into the constituents, {}, and the tautological introduction of components, ${}^{a}\Delta^{a}_t$, in a manner which does not induce any simplification, make this approach ad hoc.

There are frequent circumstances where the expressions k and k_t are both approximately equal to one (Carino, 1999, pp. 9-10). Under these circumstances, the excess geometric return, ${}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T}$, is approximately equal to the sum of the simple components, ${}^{a}\Delta^{a}_t$:

$${}^{g}R_{1 \text{ to } T} - {}^{g}R^{B}_{1 \text{ to } T} = \Sigma_{t=1 \text{ to } T}\, \{(k_t/k) * {}^{a}\Delta^{a}_t\} \approx \Sigma_{t=1 \text{ to } T}\, \{{}^{a}\Delta^{a}_t\}.$$

The important applaudable aspects of this arithmetic decomposition approach are that it is exact, that the constituents, $\{(k_t/k) * {}^a\Delta^a{}_t\}$, are often good approximations of their arithmetic counterparts, ${}^a\Delta^a{}_t$, and that its constituents are summed (not multiplied), over time, t, in order to obtain the excess geometric return. This latter positive property of additivity makes it easier for the accumulation of constituents to be understood intuitively. Thus, this approach (**AGAA**) is more useful to practitioners than the previous approach (**GGAA**), which employs a geometric decomposition.

One drawback of this approach (**AGAA**), which it shares with the previous approach **(GGAA)**, is its somewhat complicated inclusion of logarithmic expressions into the exact result for each constituent. However, beyond just being complicated and ad hoc, the above arithmetic decomposition has a major drawback. Because the exact constituent term, $\{(k_t/k) * {}^a\Delta^a{}_t\}$, for day t, depends on k, and because k, in turn, depends on the geometric returns, ${}^gR_{1\ to\ T}$ and ${}^gR^B{}_{1\ to\ T}$, for the whole period, from the start of day 1 to the end of day T, it follows that the constituent for day t depends, in part, on the return of the portfolio and the benchmark for days that come after day t. Thus, the constituent, $\{(k_t/k) * {}^a\Delta^a{}_t\}$, of the excess return associated with the day t is a-causal, in the sense that it is meant to describe a property associated with day t, but is dependent upon values (*i.e.*, the returns of the portfolio and benchmark for days later in the period) that do not actually become fixed till considerably after day t has passed. Accordingly, this method (**AGAA**) implies that today's constituent of the excess return cannot be calculated today, because its value depends on the portfolio's and benchmark's returns for days yet to arrive.

The First Step: The Decomposition of Excess Return in Terms of an Excess Incremental Return

The New Mirroring Approach

AGAG:

- The arithmetic excess of geometric returns as a simple direct sum of the arithmetic excess of the incremental geometric returns;

$${}^gR_{1\ to\ T} - {}^gR^B{}_{1\ to\ T} \equiv$$

$$\Sigma_{t=1\ to\ T}\ \{(1 + {}^gR_{1\ to\ t-1}) * R_t - (1 + {}^gR^B{}_{1\ to\ t-1}) * R^B{}_t\} =$$

$$\Sigma_{t=1\ to\ T}\ \{{}^a\Delta^g{}_t\},$$

$$\text{where, } {}^a\Delta^g{}_t \equiv ({}^gR_{1\ to\ t} - {}^gR_{1\ to\ t-1}) - ({}^gR^B{}_{1\ to\ t} - {}^gR^B{}_{1\ to\ t-1}),$$

$$\text{with } {}^gR_{1\ to\ t} \equiv [\Pi_{\tau=1\ to\ t}\ (1 + R_\tau)] - 1.$$

Why do the previous standard exact, geometric approaches (**GGAA**, **AGAA**) so quickly lead to severe complications, such as products of exponentials whose exponents require logarithmic coefficients or sums with a-causal coefficients? The reason is that these approaches are trying to force a square peg into a round hole. They construct the excess of the geometric (compounded) returns for a period, $(1 + {}^gR_{1\ to\ T})/(1+ {}^gR^B{}_{1\ to\ T}) - 1$ or ${}^gR_{1\ to\ T} - {}^gR^B{}_{1\ to\ T}$, out of terms incorporating ${}^a\Delta^a{}_t$, the arithmetic excess of the incremental arithmetic (uncompounded) returns for the days in the period. The more natural course is to employ as the concept at the foundation of an approach the term ${}^a\Delta^g{}_t$, the arithmetic excess of the incremental geometric (compounded) returns for a day. Then it becomes considerably easier to construct the excess geometric return for the period in a simple, exact and intuitive form. That is, the better approach is to have the elements that are doing the explaining *mirror* what is to be explained by constructing the arithmetic excess geometric return for a period out of the arithmetic excess of the incremental geometric returns for the individual days in the period. In fact, employing ${}^a\Delta^g{}_t$, instead of ${}^a\Delta^a{}_t$, is the crucial structural aspect that distinguishes the new mirroring approach from previous approaches.

Analogous to the definition of the arithmetic excess of the incremental arithmetic returns, ${}^a\Delta^a{}_t \equiv ({}^aR_{1\ to\ t} - {}^aR_{1\ to\ t-1}) - ({}^aR^B{}_{1\ to\ t} - {}^aR^B{}_{1\ to\ t-1})$, define the arithmetic excess of the incremental geometric returns, which is central to the new mirroring approach, as follows:

$${}^a\Delta^g{}_t \equiv (\ {}^gR_{1\ to\ t} - {}^gR_{1\ to\ t-1}\ ) - (\ {}^gR^B{}_{1\ to\ t} - {}^gR^B{}_{1\ to\ t-1}\ ) =$$

$$\{[\Pi_{\tau=1\ to\ t}\ (1 + R_\tau)] - [\Pi_{\tau=1\ to\ t-1}\ (1 + R_\tau)]\} -$$

$$\{[\Pi_{\tau=1\ to\ t}\ (1 + R^B{}_\tau)] - [\Pi_{\tau=1\ to\ t-1}\ (1 + R^B{}_\tau)]\}.$$

Thus, the new mirroring approach focuses upon the incremental difference of geometric returns, as opposed to the incremental difference of arithmetic returns.

The geometric return for the period, up to and including day t, is ${}^{g}R_{1\ to\ t} \equiv [\Pi_{\tau = 1\ to\ t}\ (1 + R_\tau)] - 1$. The incremental (single-day) geometric return for the day t is the difference between the geometric return for the period through day t and the geometric return for the period through day t – 1:

$$ {}^{g}R_{1\ to\ t} - {}^{g}R_{1\ to\ t-1} = \{[\Pi_{\tau = 1\ to\ t}\ (1 + R_\tau)] - 1\} - \{[\Pi_{\tau = 1\ to\ t-1}\ (1 + R_\tau)] - 1\}. $$

It is then simple to see that:

$$ {}^{g}R_{1\ to\ t} - {}^{g}R_{1\ to\ t-1} = [\Pi_{\tau = 1\ to\ t-1}\ (1 + R_\tau)] * R_t = $$

$$ (1 + {}^{g}R_{1\ to\ t-1}) * R_t. $$

This can be intuitively understood as stating that the amount of the change, ${}^{g}R_{1\ to\ t} - {}^{g}R_{1\ to\ t-1}$, in the geometric return during the incremental (single-day) period, between day t - 1 and day t, is just the single day return, R_t, of day t, applied to the proper basis, $(1+ {}^{g}R_{1\ to\ t-1})$, of what the portfolio has become by the start of day t.

It follows that:

$$ {}^{a}\Delta^{g}_{t} = (1 + {}^{g}R_{1\ to\ t-1}) * R_t - (1 + {}^{g}R^{B}_{1\ to\ t-1}) * R^{B}_{t}. $$

(This result should be compared to the arithmetic one, ${}^{a}\Delta^{a}_{t} = R_t - R^{B}_{t}$.)

Defining ${}^{g}R^{B}_{1\ to\ 0} = 0$, the return accumulated over a period T is then obtained by summing the daily changes over all days in the period:

$$ {}^{g}R_{1\ to\ T} = \Sigma_{t = 1\ to\ T}\ [{}^{g}R_{1\ to\ t} - {}^{g}R_{1\ to\ t-1}] = $$

$$ \Sigma_{t = 1\ to\ T}\ [(1 + {}^{g}R_{1\ to\ t-1}) * R_t].^{7} $$

(Again, an informative comparison is to the arithmetic result, ${}^{a}R_{1\ to\ T} = \Sigma_{t = 1\ to\ T}\ [{}^{a}R_{1\ to\ t} - {}^{a}R_{1\ to\ t-1}] = \Sigma_{t = 1\ to\ T}\ [R_t]$.)

Thus:

$$ {}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T} \equiv $$

$$ \Sigma_{t = 1\ to\ T}\ [(1 + {}^{g}R_{1\ to\ t-1}) * R_t - (1 + {}^{g}R^{B}_{1\ to\ t-1}) * R^{B}_{t}] = $$

$$ \Sigma_{t = 1\ to\ T}\ \{{}^{a}\Delta^{g}_{t}\}. $$

This successfully completes the first step of obtaining a simple and intuitive summation as the expression of the decomposition of the arithmetic excess of the geometric returns for the period into the arithmetic excess of the incremental geometric returns.

This result should be compared with those obtained in the initial steps of the previous approaches to decomposing the excess geometric return into analogous constituent terms:

$$ (1 + {}^{g}R_{1\ to\ T})/(1 + {}^{g}R^{B}_{1\ to\ T}) = $$

$$ \Pi_{t = 1\ to\ T}\ \{\exp[\ln(1 + R_t) - \ln(1+ R^{B}_{t})]\}, $$

for the geometric excess of geometric returns (**GGAA**), and

$$ {}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T} = $$

$$ \Sigma_{t = 1\ to\ T}\ \{[({}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T})/ $$

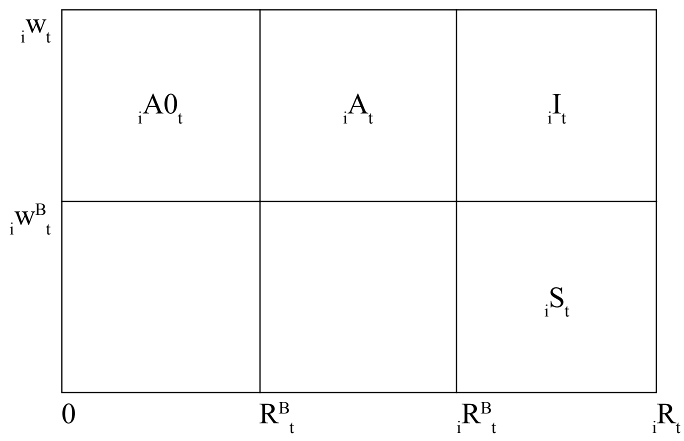
$$ (\ln(1 + {}^{g}R_{1\ to\ T}) - \ln(1 + {}^{g}R^{B}_{1\ to\ T}))] * $$

$$ [\ln(1 + R_t) - \ln(1 + R^{B}_{t})]\}, $$

for the arithmetic excess of geometric returns in the improved approach (**AGAA**). (Further on, Table 3a (*see page* 00) will exhibit a numerical application of these three equations, demonstrating the intuitiveness of the new mirroring approach.)

All three decompositions of the excess geometric return in terms of daily returns are exactly mathematically correct. However, even at the end of this initial step of decomposition, it is seen that the standard geometric approach (**GGAA**) explains via a product instead of a sum, and the improved arithmetic approach (**AGAA**) is a-causal. Both employ logarithmic functions. Consequently, these three (**GGAA**, **AGAA**, **AGAG**) different approaches do not all lead to equally successful ultimate decompositions. It is also important that neither of the standard decompositions (**GGAA**, **AGAA**) result in constituents, which have any natural relationship to the components of the excess arithmetic incremental

Figure 1
The Arithmetic PAR (AAAA).



return, ${}^a\Delta^a_t$, which they wish to incorporate. That is why they have to introduce it by fiat through the insertion of a tautological expression (multiplication by "1" where $1 = \{{}^a\Delta^a_t/[R_t - R^B_t]\}$) which involves the desired term, ${}^a\Delta^a_t$, and then obtain no simplification by cancellation.

In a manner analogous to the standard approaches (**AAAA**, **GGAA**, and **AGAA**), the new mirroring approach (**AGAG**) decomposes the excess geometric return, ${}^gR_{1\text{ to }T} - {}^gR^B_{1\text{ to }T}$, into individual constituents in curly brackets, {}, which are associated with components, ${}^a\Delta^g_t$, of the excess return. However, in this new mirroring case, the incorporation of the desired component did not occur by tautological fiat without simplification, but rather, it was introduced through a natural substitution, as in the original arithmetic approach (**AAAA**).

This first step of the new mirroring approach to the decomposition of the excess geometric return has a number of important properties. The result, while being exact, is considerably simpler than that obtained by the previous exact approaches, because the excess geometric return, ${}^gR_{1\text{ to }T} - {}^gR^B_{1\text{ to }T}$, is decomposed into the simple direct sum of the components, ${}^a\Delta^g_t$, which will, in the second step, be resolved into the sum of contributions of the decision types. Its constituents, $\{{}^a\Delta^g_t\}$, are not a-causal. The result of accumulation also makes intuitive sense since the constituents, $\{{}^a\Delta^g_t\}$, are simple to understand as the actual components, ${}^a\Delta^g_t$, of the arithmetic excess incremental geometric return themselves, instead of being a complicated function of the components, ${}^a\Delta^g_t$, of the arithmetic excess of the incremental arithmetic returns. Finally, at each step the results of **AGAG** are good approximations of their arithmetic (**AAAA**) counterparts when compounding is negligible.

The Second Step: The Decomposition of Incremental Returns and its Visualization

Three Previous Approaches

The second step in the explication of the structure of performance attribution is to decompose the excess incremental return in terms of the contributions of decision types.

AAAA:

The historically original scheme (**AAAA**) (Brinson, 1985, 1986, and 1991; Karnosky, 1994; and Singer, 1995) decomposes the arithmetic excess of the incremental arithmetic returns as follows:

$$R_t - R^B_t = {}^a\Delta^a_t = \text{Allocation (t)} + \text{Selection (t)} +$$

$$\text{Interaction (t)} = \Sigma_i\{_iA_t\} + \Sigma_i\{_iS_t\} + \Sigma_i\{_iI_t\},$$

$$\text{or, } \Sigma_i[_iw_t * {}_iR_t - {}_iw^B_t * {}_iR^B_t] = \{\Sigma_i[(_iw_t - {}_iw^B_t) *$$

$$(_iR^B_t - R^B_t)]\} + \{\Sigma_i[_iw^B_t * (_iR_t - {}_iR^B_t)]\} + \{\Sigma_i[(_iw_t - {}_iw^B_t)$$

$$* (_iR_t - {}_iR^B_t)]\}$$

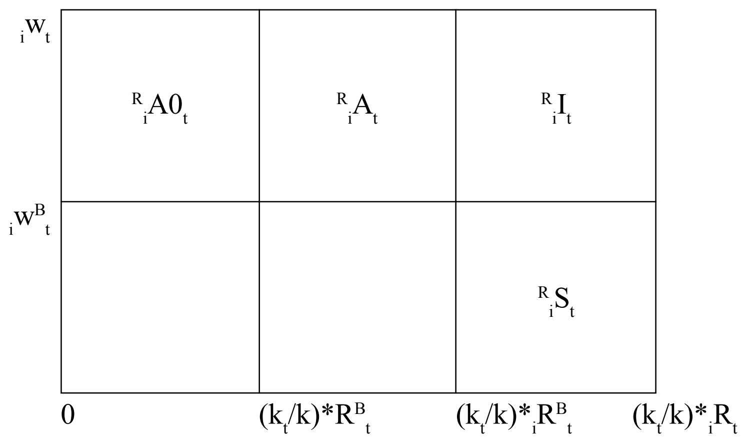
(see Appendix *2i* on page 00).

The index "i" scans some segmentation of the investment universe (here called "industries") such that the portfolio return for a day is the weighted sum of the returns of the segments on that day:

$$R_t = \Sigma_i(_iw_t * {}_iR_t).$$

The term $_iw_t$ represents the weight of the industry in the portfolio at the start of day t, and the term $_iR_t$ represents the return of industry i for the day t. The decisions on day t are here taken to be allocation, selection and interaction. They each, in turn, are decomposed into the allocation, selection and interaction decisions associated with each industry i on that day t.

Figure 2
The Carino/Russell PAR (AGAA).

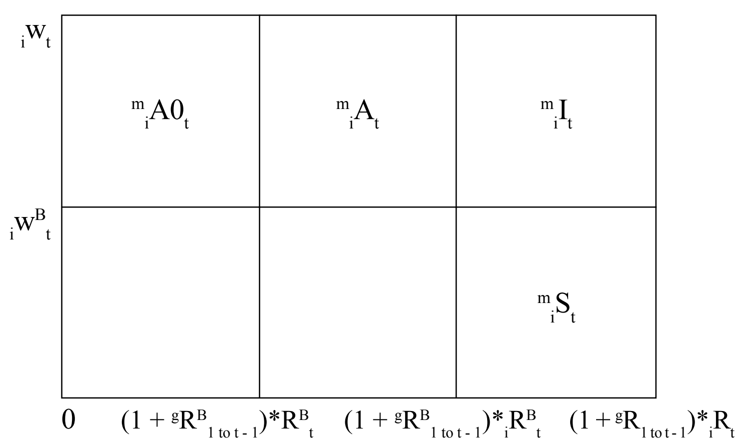


The crucial insight regarding this decomposition is that it is not a list of contributions that just happen to add up to the desired result, the arithmetic excess of the incremental arithmetic returns, and which it is standard to represent as an unstructured concatenation of Quadrants (Brinson, 1985, 1986, and 1991; Karnosky, 1994; and Singer, 1995). Rather, the decomposition directly follows from a geometric analogy (Fisher, 1985, pp. 2) that can be visualized as the sum, spanning industries, of the areas of sub-rectangles of a Performance Attribution Rectangle, (hereafter referred to as a PAR), where the areas of the sub-rectangles are obtained by multiplying their bases and heights as given by their coordinates. The very structure of a PAR incorporates the full definitions of the contributions, which compose the excess incremental return.

The arithmetic PAR in Figure 1 shows that the area of the rectangle formed by the product of the portfolio's industry weight, ${}_iw_t$, and portfolio's industry return, ${}_iR_t$, minus the area formed by the product of the benchmarks industry weight, ${}_iw^B_t$, and the benchmark's industry return, ${}_iR^B_t$, is equal to the sum of the areas ${}_iA_t + {}_iS_t + {}_iI_t$ (contributions due to industry level allocation, selection and interaction) plus an area, ${}_iA0_t$. However, area ${}_iA0_t$ can be neglected since adding together its values for all industries, for any day t, gives zero: $\Sigma_i\, {}_iA0_t = 0$.[8] Thus, it is seen that attribution calculations can be made transparent by consideration of the information provided by the PARs, especially by reading the definitions of the contributions off of (the areas of) the sub-rectangles of a PAR. For example, the contribution of the allocation, ${}_iA_t$, of industry i on day t for arithmetic attribution **(AAAA)** is just the area of the corresponding sub-rectangle of the arithmetic PAR. This area is the product of its height and its base as obtained from its given coordinates on the PAR (Figure 1):

$$ {}_iA_t = ({}_iw_t - {}_iw^B_t)*({}_iR^B_t - R^B_t). $$

Figure 3
The New Mirroring PAR (AGAG).



If the designated sub-rectangles of each arithmetic PAR are simply summed over time and over industry one obtains:

$$ \Sigma_{t=1\text{ to }T,\,i}\,\{{}_iA_t + {}_iS_t + {}_iI_t\} = $$

$$ \Sigma_{t=1\text{ to }T,\,i}\,\{{}_iw_t * {}_iR_t - {}_iw^B_t * {}_iR^B_t\} = $$

$$ \Sigma_{t=1\text{ to }T}\,\{R_t - R^B_t\} = {}^aR_{1\text{ to }T} - {}^aR^B_{1\text{ to }T}, $$

which is simply the arithmetic excess of the arithmetic returns. Recall that the historically original method (**AAAA**), here represented by the arithmetic PAR of Figure 1, ignores completely compounding completely.

GGAA:

The single day t attribution effects, ${}_iA_t$, ${}_iS_t$, and ${}_iI_t$, can be combined to obtain the geometric excess of the geometric returns of the whole period, T, (Carino, 1999, pp. 9) by multiplying the sum of these effects by the log factor k_t, and then taking the exponential and multiplying the result over time and industry so that the following decomposition is obtained:

$$(1 + {}^gR_{1\ to\ T})/(1+ {}^gR^B_{1\ to\ T}) =$$

$$\Pi_{t=1\ to\ T}\ \{\exp[(1 + R_t)/(1 + R^B_t)]\} =$$

$$\Pi_{t=1\ to\ T}\ \{\exp(k_t * {}^a\Delta^a_t)\},$$

$$\text{or, } (1 + {}^gR_{1\ to\ T})/(1+ {}^gR^B_{1\ to\ T}) =$$

$$\Pi_{t=1\ to\ T,\ i}\ \{\exp[k_t * ({}_iA_t + {}_iS_t + {}_iI_t)]\} =$$

$$\Pi_{t=1\ to\ T,\ i}\ \{(1 + {}^\pi_iA_t) * (1 + {}^\pi_iS_t) * (1+ {}^\pi_iI_t)]$$

(See Appendix *2ii* on page 00).

This multiplicative approach (**GGAA**) to performance attribution cannot be visualized by a PAR.

AGAA:

The single day t, industry-specific, attribution effects, ${}_iA_t$, ${}_iS_t$, and ${}_iI_t$, can be combined to obtain the arithmetic excess of the geometric returns of the whole period, T, (Carino, 1999, pp. 8) by multiplying each of these effects by the log factor: k_t/k, and then summing the result over time and industry so that the following decomposition is obtained:

$${}^gR_{1\ to\ T} - {}^gR^B_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ \{(k_t/k) * (R_t - R^B_t)\} =$$

$$\Sigma_{t=1\ to\ T}\ \{(k_t/k) * {}^a\Delta^a_t\} =$$

$$\Sigma_{t=1\ to\ T,\ i}\ \{(k_t/k) * ({}_iA_t + {}_iS_t + {}_iI_t)\},$$

$$\text{or } {}^gR_{1\ to\ T} - {}^gR^B_{1\ to\ T} = \Sigma_{t=1\ to\ T,\ i}\ \{{}^R_iA_t + {}^R_iS_t + {}^R_iI_t\}$$

(See Appendix *2v* on page 00).

This approach (**AGAA**) can be visualized by the PAR shown in Figure 2. The dependence upon k makes the daily terms a-causal, since k depends upon period-wide results.

The Second Step: The Decomposition of Incremental Returns and its Visualization

The New Mirroring Approach

AGAG:

The simple new mirroring alternative (**AGAG**) to these previous approaches (**AAAA**, **GGAA**, **AGAA**) is to decompose the arithmetic excess of the incremental geometric returns by directly employing the PAR shown in Figure 3. As in the arithmetic (**AAAA**) and improved (**AGAA**) approaches, for **AGAG** there is one PAR for each industry on each day.

As follows from the calculation in Appendix 1 (*see page* 00), this decomposition (**AGAG**) can be written:

$${}^gR_{1\ to\ T} - {}^gR^B_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ \{{}^a\Delta^g_t\} =$$

$$\Sigma_{t=1\ to\ T,\ i}\ \{(1 + {}^gR_{1\ to\ t-1}) * ({}_iw_t * {}_iR_t) -$$

$$(1 + {}^gR^B_{1\ to\ t-1}) * ({}_iw^B_t * {}_iR^B_t)\}.$$

In a construction parallel to the presentation of the previous approaches, as put forth in the preceding section, an individual term can be represented by the mirroring PAR of Figure 3, and each such PAR can be analyzed:

$$\{(1 + {}^gR_{1\ to\ t-1}) * ({}_iw_t * {}_iR_t) - (1 + {}^gR^B_{1\ to\ t-1}) * ({}_iw^B_t * {}_iR^B_t)\} =$$

$${}^m_iA_t + {}^m_iS_t + {}^m_iI_t.$$

Then the straightforward sum over industries and days of the various contributions of this decomposition directly gives the arithmetic excess of the geometric returns:

Table 1
Numerical Comparison of Contributions Where Compounding Effects Are Small.

		Input Weights and Returns				First-period Contributions to 3 Periods				Rebalanced 3-Period Contributions to 3 Periods			
AAAA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0008	-0.0060	-0.0010	-0.0062	0.0024	-0.0180	-0.0030	-0.0186
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0024	0.0060	-0.0030	0.0054	0.0072	0.0180	-0.0090	0.0162
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0022	0.0000	0.0010	-0.0012	-0.0066	0.0000	0.0030	-0.0036
	Total	1.0000	0.0700	1.0000	0.0720	0.0010	**0.0000**	-0.0030	-0.0020	0.0030	**0.0000**	-0.0090	-0.0060
				R-Rb=	-0.0020							3*R-3*Rb=	-0.0060
GGAA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0007	-0.0056	-0.0009	-0.0058	0.0022	-0.0167	-0.0028	-0.0172
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0022	0.0056	-0.0028	0.0051	0.0067	0.0169	-0.0084	0.0152
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0021	0.0000	0.0009	-0.0011	-0.0061	0.0000	0.0028	-0.0034
	Total	1.0000	0.0700	1.0000	0.0720	0.0009	**0.0000**	-0.0028	-0.0019	0.0028	**0.0000**	-0.0084	-0.0056
			(1+R)/(1+Rb)-1=		-0.0019						(1+R3)/(1+Rb3)-1=		-0.0056
		R1=	0.0700	Rb1=	0.0720	k1=	0.9337						
		R2=	0.1449	Rb2=	0.1492								
		R3=	0.2250	Rb3=	0.2319								
GGGA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0007	-0.0056	?	?	?	?	?	?
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0022	0.0056	?	?	?	?	?	?
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0021	0.0000	?	?	?	?	?	?
	Total	1.0000	0.0700	1.0000	0.0720	0.0009	**0.0000**	-0.0028	-0.0019	0.0028	**0.0000**	-0.0084	-0.0056
											(1+R3)/(1+Rb3)-1=		-0.0056
GGGA2		W	R	Wb	Rb	Allocation	Selection & Interaction		Total	Allocation	Selection & Interaction		Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0007	-0.0065		?	?	?		?
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0022	0.0028		?	?	?		?
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0021	0.0009		?	?	?		?
	Total	1.0000	0.0700	1.0000	0.0720	0.0009	-0.0028		-0.0019	0.0028	-0.0084		-0.0056
											(1+R3)/(1+Rb3)-1=		-0.0056
AGAA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0009	-0.0069	-0.0011	-0.0071	0.0028	-0.0206	-0.0034	-0.0213
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0028	0.0069	-0.0034	0.0062	0.0083	0.0206	-0.0103	0.0186
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0025	0.0000	0.0011	-0.0014	-0.0076	0.0000	0.0034	-0.0041
	Total	1.0000	0.0700	1.0000	0.0720	0.0011	**0.0000**	-0.0034	-0.0023	0.0034	**0.0000**	-0.0103	-0.0069
				R-Rb=	-0.0020							R3-Rb3=	-0.0069
						k1=	0.9337			k =	0.8140		
AGAA2		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0008	-0.0060	-0.0010	-0.0062	0.0024	-0.0179	-0.0032	-0.0187
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0024	0.0060	-0.0030	0.0054	0.0072	0.0181	-0.0097	0.0157
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0022	0.0000	0.0010	-0.0012	-0.0066	0.0000	0.0028	-0.0038
	Total	1.0000	0.0700	1.0000	0.0720	?	?	?	?	0.0030	0.0002	-0.0101	-0.0069
												R3-Rb3=	-0.0069
AGAG		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.7000	0.0700	0.6000	0.0800	0.0008	-0.0060	-0.0010	-0.0062	0.0026	-0.0196	-0.0033	-0.0203
	Bonds	0.2000	0.0750	0.4000	0.0600	0.0024	0.0060	-0.0030	0.0054	0.0077	0.0191	-0.0096	0.0173
	Cash	0.1000	0.0600	0.0000	0.0500	-0.0022	0.0000	0.0010	-0.0012	-0.0071	0.0000	0.0032	-0.0039
	Total	1.0000	0.0700	1.0000	0.0720	0.0010	**0.0000**	-0.0030	-0.0020	0.0032	**-0.0005**	-0.0097	-0.0069
												R3-Rb3=	-0.0069

Table 2
Numerical Comparison of Contributions Where Compounding Effects Are Large.

		Input Weights and Returns				First-period Contributions to 3 Periods				Rebalanced 3-Period Contributions to 3 Periods			
AAAA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0252	0.0054	-0.0014	-0.0212	-0.0756	**0.0162**	-0.0042	-0.0636
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0360	-0.0120	-0.0240	-0.0720	-0.1080	-0.0360	-0.0720	-0.2160
	Cash	0.3000	0.0700	0.3600	0.0100	0.0132	0.0216	-0.0036	0.0312	0.0396	0.0648	-0.0108	0.0936
	Total	1.0000	0.1680	1.0000	0.2300	-0.0480	0.0150	-0.0290	-0.0620	-0.1440	0.0450	-0.0870	-0.1860
				R-Rb=	-0.0620							3*R-3*Rb=	-0.1860
GGAA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0208	0.0045	-0.0012	-0.0175	-0.0611	**0.0136**	-0.0035	-0.0517
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0296	-0.0100	-0.0198	-0.0583	-0.0862	-0.0296	-0.0583	-0.1649
	Cash	0.3000	0.0700	0.3600	0.0100	0.0111	0.0182	-0.0030	0.0264	0.0336	0.0555	-0.0090	0.0812
	Total	1.0000	0.1680	1.0000	0.2300	-0.0393	0.0126	-0.0239	-0.0504	-0.1132	0.0383	-0.0700	-0.1437
				(1+R)/(1+Rb)-1=	-0.0504							(1+R3)/(1+Rb3)-1=	-0.1437
		R1=	0.1680	Rb1=	0.2300	k1=	0.8342						
		R2=	0.3642	Rb2=	0.5129								
		R3=	0.5934	Rb3=	0.8609								
GGGA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0205	0.0044	?	?	?	?	?	?
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0293	-0.0098	?	?	?	?	?	?
	Cash	0.3000	0.0700	0.3600	0.0100	0.0107	0.0176	?	?	?	?	?	?
	Total	1.0000	0.1680	1.0000	0.2300	-0.0390	0.0122	-0.0237	-0.0504	-0.1126	0.0370	-0.0696	-0.1437
												(1+R3)/(1+Rb3)-1=	-0.1437
GGGA2		W	R	Wb	Rb	Allocation	Selection & Interaction		Total	Allocation	Selection & Interaction		Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0205	0.0034		?	?	?		?
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0293	-0.0305		?	?	?		?
	Cash	0.3000	0.0700	0.3600	0.0100	0.0107	0.0152		?	?	?		?
	Total	1.0000	0.1680	1.0000	0.2300	-0.0390	-0.0118		-0.0504	-0.1126	-0.0351		-0.1437
												(1+R3)/(1+Rb3)-1=	-0.1437
AGAA		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0362	0.0078	-0.0020	-0.0305	-0.1087	**0.0233**	-0.0060	-0.0915
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0518	-0.0173	-0.0345	-0.1035	-0.1553	-0.0518	-0.1035	-0.3106
	Cash	0.3000	0.0700	0.3600	0.0100	0.0190	0.0311	-0.0052	0.0449	0.0569	0.0932	-0.0155	0.1346
	Total	1.0000	0.1680	1.0000	0.2300	-0.0690	0.0216	-0.0417	-0.0892	-0.2071	0.0647	-0.1251	-0.2675
				R-Rb=	-0.0620							R3-Rb3=	-0.2675
						k1=	0.8342			k =	0.5802		
AGAA2		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0252	0.0054	-0.0014	-0.0212	-0.0737	**0.0163**	-0.0087	-0.0661
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0360	-0.0120	-0.0240	-0.0720	-0.1042	-0.0356	-0.1456	-0.2854
	Cash	0.3000	0.0700	0.3600	0.0100	0.0132	0.0216	-0.0036	0.0312	0.0401	0.0662	-0.0223	0.0840
	Total	1.0000	0.1680	1.0000	0.2300	?	?	?	?	-0.1377	0.0469	-0.1766	-0.2675
												R3-Rb3=	-0.2675
AGAG		W	R	Wb	Rb	Allocation	Selection	Interaction	Total	Allocation	Selection	Interaction	Total
	Equities	0.4000	0.4200	0.5400	0.4100	-0.0252	0.0054	-0.0014	-0.0212	-0.0943	**-0.0276**	0.0071	-0.1147
	Bonds	0.3000	-0.0700	0.1000	0.0500	-0.0360	-0.0120	-0.0240	-0.0720	-0.1347	-0.0434	-0.0869	-0.2651
	Cash	0.3000	0.0700	0.3600	0.0100	0.0132	0.0216	-0.0036	0.0312	0.0494	0.0755	-0.0126	0.1124
	Total	1.0000	0.1680	1.0000	0.2300	-0.0480	0.0150	-0.0290	-0.0620	-0.1797	0.0045	-0.0923	-0.2675
												R3-Rb3=	-0.2675

Table 3a
The Importance of the Basis for the Incremental Excesses.

t	R	1+R	$1+{}^{g}R$	MV	t	R^B	$1+R^B$	$1+{}^{g}R^B$	MV^B
	Portfolio Return for Day t.			Portfolio Market Value at Close of Day t.		Benchmark Return for Day t.			Benchmark Market Value at Close of Day t.
0		1.00	1.00	$100.00	0		1.0	1.00	$100.00
1	**15%**	1.15	1.15	$115.00	1	**10%**	1.1	1.10	$110.00
2	-60%	0.40	0.46	$46.00	2	10%	1.1	1.21	$121.00
3	**15%**	1.15	0.53	$52.90	3	**10%**	1.1	1.33	$133.10

t	Excess Arithmetic Return for Day t. $R - R^B$	Excess Geometric Return for Day t. ${}^{g}R - {}^{g}R^B$	Incremental Excess Geometric Return for Day t. $D({}^{g}R - {}^{g}R^B)$	Excess Market Value at Close of Day t. $MV - MV^B$	Incremental Excess Market Value at Close of Day t. $D(MV - MV^B)$
0	0.00%	0.00%	0.00%	$0.00	$0.00
1	**5.00%**	5.00%	**5.00%**	$5.00	**$5.00**
2	-70.00%	-75.00%	-80.00%	-$75.00	-$80.00
3	**5.00%**	-80.20%	**-5.20%**	-$80.20	**-$5.20**
sum =	-60.00%		-80.20%		-$80.20
	$(1 + {}^{g}R)/(1 + {}^{g}R^B) - 1$		-60.26%		

For every $100 the portfolio started with it lost $80.20 relative to the benchmark over the total period. This loss came from a $5.00 relative gain on the first day; an $80.00 relative loss on the second day; and a $5.20 relative loss on the third day.

$${}^{g}R_{1\text{ to }T} - {}^{g}R^B_{1\text{ to }T} = \Sigma_{t=1\text{ to }T,\,i}\ \{{}^{m}_{i}A_t + {}^{m}_{i}S_t + {}^{m}_{i}I_t\}.$$

(See Appendix 1 on page 00 for details of the calculation.)

The visualization, by means of PARs, of the new mirroring geometric approach (**AGAG** and Figure 3) demonstrates its simplicity, while making obvious the intuitive internal logic for compounding attribution effects. Unlike the situation with the previous improved approach (**AGAA**), the new mirroring approach (**AGAG**) does not just arbitrarily "distribute the error in the (arithmetic) approximation among the effects" (Carino, 1999, pp. 9). Rather, the contributions are defined in a manner that gives them a clear intuitive meaning. Unlike the improved approach (**AGAA**), the new mirroring approach (**AGAG**) is not a-causal. Furthermore, the new mirroring approach (**AGAG**) preserves the important additivity property in that it is the simple direct sum of the contributions that correctly give the exact arithmetic excess of the geometric returns.

As an example of how the new mirroring PAR correctly informs the intuition, consider the explication of the contribution, ${}^{m}_{i}S_t$, to the arithmetic excess geometric return, ${}^{g}R_{1\text{ to }T} - {}^{g}R^B_{1\text{ to }T}$, due to stock selection of industry i on day t. As can be read off the PAR of Figure 3, ${}^{m}_{i}S_t$ is equal to the benchmark weight, ${}_{i}w^B_t$, of the industry at the start of the day multiplied by an incremental

Table 3b
The Components Proposed by the Different Approaches for Explaining the Excess Incremental Returns

t	AAAA $R - R^B$	GGAA & GGGA $(1+R)/(1+R^B) - 1$	AGAA $(k_t/k) * (R - R^B)$	AGAG $((1+{}^gR_{t-1}) * R - (1+{}^gR^B_{t-1}) * R^B)$
0				
1	5.00%	4.55%	3.86%	**5.00%**
2	-70.00%	-63.64%	-87.93%	-80.00%
3	5.00%	4.55%	3.86%	**-5.20%**
sum=	-60.00%		-80.20%	-80.20%
	$\{\Pi_t[(1+R)/(1+R^B) - 1]\}$	-60.26%		

While the components of both **AGAA** and **AGAG** match the change in the relative Market Value for the whole period, only **AGAG**'s components match the daily change in relative Market Value and geometrically accumulated excess returns, as given in Appendix 5a.

difference term, $\{(1+{}^gR_{1\,to\,t-1}) * {}_iR_t - (1+{}^gR^B_{1\,to\,t-1}) * {}_iR^B_t\}$, which is simply the portfolio's industry return on the amount by which the portfolio has grown, during the period up till day t, minus the benchmark's industry return on the amount by which the benchmark has grown. This weighted excess captures the intuition that the greater the difference between the amount that the industry causes the portfolio to grow and the amount that the industry causes the benchmark to grow, the more the industry contributes to the excess geometric return for the period. See Appendix 2 (*see page* 00) for a full comparison of the algebraic form of the contributions $\{{}^m_iA_t, {}^m_iS_t, {}^m_iI_t\}$ and their concatenations in the new mirroring approach (**AGAG**) to those of other approaches (**AAAA**, **GGAA**, **GGGA**, **GGGA2**, **AGAA**, and **AGAA2**) to performance attribution.

It is worth mentioning that for none of the stated approaches is it true that an individual industry's allocation (or selection, or interaction) defined for the weights and returns for a period taken as a whole, identically equals the combination of that industry's allocations of the sub-periods that comprise the whole period.[9] This suggests that the allocation for the whole period should, for consistency, always be considered to be the result derived from the set of the most atomic periods considered. The frequency of evaluating the price of the whole portfolio puts a lower limit on how short the most atomic time period can be.

Additionally, for the only two complete cases (**AGAA** and **AGAG**) that explain the arithmetic excess of geometric returns exactly, today's contribution to the month is not today's contribution to the year.

NUMERICAL COMPARISONS

All discussed approaches give approximately the same results within the limits in which we would expect them to agree and in which we would expect our intuitions to be reliable. These are situations in which compounding can be ignored, since the value of the portfolio and benchmark do not change significantly during a single day. Table 1 (*see page* 00) compares the values of the contributions obtained by each of the approaches for a numerical example in which compounding is negligible (Carino, 1999).[10] Table 2 (*see page* 00) compares the values of the same approaches for a case with larger returns, making compounding more significant. As would be expected, in this second case the comparison shows that the results of the different approaches are quite dissimilar (equity selection even changes sign), thus forcing a practical choice between the different approaches.

One structural difference between the new mirroring approach (**AGAG**) and some of the other approaches

(**AAAA**, **GGAA**, **GGGA**, and **AGAA**) will be pointed out. For the numerical example of Table 1 (*see page* 00), these approaches have zero total selection for the first sub-period. The two subsequent sub-periods replicate the weights and returns of the first period. In this case, all the other such approaches obtain zero selection for these sub-periods also, and thus, for the total period. However, the new mirroring approach (**AGAG**) obtains nonzero selection for these subsequent periods because the returns of these periods are creating wealth starting from a different basis. That is, due to the change, from day to day, of the difference between the portfolio's geometric factor and the benchmark's geometric factor, $(1 + {}^{g}R_{1\,to\,2-1}) - (1 + {}^{g}R^{B}_{1\,to\,2-1}) \neq (1 + {}^{g}R_{1\,to\,1-1}) - (1 + {}^{g}R^{B}_{1\,to\,1-1})$, the selection can change from day to day even if the weights and returns themselves do not change from day to day. In other words, how much a decision regarding a portfolio contributes on a day to a total period depends on the amount to which the portfolio grew relative to the benchmark in the previous days. If the portfolio previously lost a great deal of its capital, a subsequent very good decision still might not be good enough to contribute positively to the excess return.[11]

As a simple buy-and-hold example of the importance of using the correct basis, consider the case, exhibited in Table 3a (*see page* 00). A portfolio starts with \$100 and returns 15%, –60%, and 15% on three subsequent days. Simultaneously, the benchmark (normalized and without cash flows) starts with \$100 and returns 10% on each of these three days. On the first day the portfolio outperforms the benchmark by $R_1 - R_1^B = 15\% - 10\% = 5\%$, while increasing its market value over the benchmark by \$5.00. On the third day the portfolio again outperforms the benchmark by 5%, but on this day the portfolio decreases its market value relative to the benchmark by \$5.20. As it is the purpose of performance attribution to indicate the reasons that \$5.00 was gained relative to the benchmark on the first day, it is also its purpose to explain the reason that \$5.20 was lost relative to the benchmark on the third day, despite the fact that the portfolio outperformed the benchmark on this third day. The explanation is the low portfolio basis at the start of the third day, brought on by the large portfolio loss on the second day. Table 3b (*see page* 00) exhibits the values of the contributions according to the different approaches. These contributions are calculated by applying the equations obtained at the end of the first step, which decompose the excess return for the period into their corresponding single-day constituents. It is seen that the new mirroring approach (**AGAG**) is the only one for which the single-day constituents of the excess return match the arithmetic incremental change in the excess period-to-date compounded returns, $\Delta({}^{g}R_{1\,to\,T} - {}^{g}R^{B}_{1\,to\,T})$ and, a fortiori, match the change in the excess market values on a daily basis. (Of course, **GGAA** and **GGGA** are not even aiming to match an arithmetic excess.) The upshot of this example is not affected by the injection of large cash flows, since the benchmark holdings on a day would then be scaled to keep them in line with such changes in the portfolio holdings.

EXTENSIONS

This new mirrored decomposition method (**AGAG**) can be further developed to include the insights of Brinson Partners (Karnosky, 1994, 1995, and Singer, 1998) regarding currency effects and hedging. It can also be extended to include other important concerns of performance attribution such as inter-day trades, and/or to be applicable at the issue level, in a manner formally analogous, and appropriately related, to the method explicated above for the industry level. At the issue level one obtains, both for a single day and for the whole period, the individual issue's allocation, selection, and interaction contribution to the arithmetic excess of the geometric returns of the whole portfolio. The new mirroring method (**AGAG**) can also be applied simultaneously at a number of levels to give, for example, sector selection, industry selection within sectors, and stock selection within industries, in a manner which sums up to the excess geometric return for the total of many concatenated periods. (See Bodie, 1986, pp. 793-797, for a single period example of this cascading multilevel approach.) The method (**AGAG**) can be further developed to separate out the components of compounding from the components attributable to the decisions of each day in isolation, for each type of contribution at the issue, industry and portfolio level.[12] In all variations of the new mirroring approach (**AGAG**), the PARs can always be directly summed to produce the exact excess geometric return. These PARs can be visualized to make the logic of performance attribution intuitively clear and, thus, lead to still other important insights into the attribution of performance.

CONCLUSION

The new mirroring approach (**AGAG**) to performance attribution developed herein is exact, simply additive, timely, natural, easily visualized, and general. It exactly decomposes the excess compounded return for an extended period of a portfolio over its benchmark, into a simple sum of contributions, at the daily and/or industry level, individually attributable to the allocation, selection and interaction effects which comprise it, without introducing any a-causal elements and where each contribution is derived by taking account of compounding. This decomposition is achieved quite naturally by relying upon the excess of the incremental geometric returns instead of the excess of the incremental arithmetic returns. Thus, the arithmetic excess of the geometric returns for a period is decomposed into elements, which logically mirror its structure.

The visualization of a decomposition by means of PARs (Figures 1, 2, and 3 (*see pages* 00-00)) demonstrates its simplicity, and makes its internal logic intuitively clear. In addition, employing PARs facilitates the comparison of various approaches to performance attribution, such as Brinson's uncompounded attribution (**AAAA**), the a-causal log-factor approach (**AGAA**) of Carino/ Frank Russell Company, and the new mirroring approach described in this paper (**AGAG**). This new mirroring solution (**AGAG**) is directly extendable, in a manner that can also be illustrated by PARs, to the full country/ currency decompositions expounded by Brinson Partners, to the inclusion of inter-day transactions, as well as to analysis at the issue level and to a combination of sector and industry (and issue) levels.

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APPENDIX 1

The detailed decomposition for the new mirroring approach (**AGAG**)

Starting from the results of the section on the First Step on **AGAG**, the calculation of the new mirroring approach (**AGAG**) goes as follows:

$$
\begin{aligned}
{}^{g}R_{1\text{ to }t} - {}^{g}R^{B}_{1\text{ to }t} &= \Sigma_{t=1\text{ to }T}\ \{(1 + {}^{g}R_{1\text{ to }t-1}) * R_t - (1 + {}^{g}R^{B}_{1\text{ to }t-1}) * R^{B}_{t}\} \\
&= \Sigma_{t=1\text{ to }T}\ \{(1 + {}^{g}R_{1\text{ to }t-1}) * \Sigma_i({}_iw_t * {}_iR_t) - (1 + {}^{g}R^{B}_{1\text{ to }t-1}) * \Sigma_i({}_iw^{B}_{t} * {}_iR^{B}_{t})\} \\
&= \Sigma_{t=1\text{ to }T}\ \Sigma_i\{(1 + {}^{g}R_{1\text{ to }t-1}) * ({}_iw_t * {}_iR_t) - (1 + {}^{g}R^{B}_{1\text{ to }t-1}) * ({}_iw^{B}_{t} * {}_iR^{B}_{t})\}.
\end{aligned}
$$

$$
{}^{g}R_{1\text{ to }t} - {}^{g}R^{B}_{1\text{ to}} = \Sigma_{t=1\text{ to }T,\, i}\ \{(1 + {}^{g}R_{1\text{ to }t-1}) * ({}_iw_t * {}_iR_t) - (1 + {}^{g}R^{B}_{1\text{ to }t-1}) * ({}_iw^{B}_{t} * {}_iR^{B}_{t})\}.
$$

Thus, the arithmetic excess of the geometric returns is just the sum, over days and over industries, of the named rectangles, ${}^{m}_{i}A0_t$, ${}^{m}_{i}A_t$, ${}^{m}_{i}S_t$ and ${}^{m}_{i}I_t$, of the new mirroring PAR (Figure 3 (*see page* 00)).

Consider: $\Sigma_i\{{}^{m}_{i}A0_t\} = \Sigma_i\{({}_iw_t - {}_iw^B_t) * (1 + {}^gR^B_{1\ to\ t-1}) * R^B_t\}$

$$= (1 + {}^gR^B_{1\ to\ t-1}) * \Sigma_i\{ R^B_t * ({}_iw_t - {}_iw^B_t) \} = (1 + {}^gR^B_{1\ to\ t-1}) * \Sigma_i\{{}_iA0_t\}$$

$$= [(1 + {}^gR^B_{1\ to\ t-1}) * R^B_t * \{[\Sigma_i({}_iw_t)] - [\Sigma_i({}_iw^B_t)]\}]$$

$$= [(1 + {}^gR^B_{1\ to\ t-1}) * R^B_t * \{1 - 1\}] = 0.$$

Thus, the upper left-hand rectangles of the new mirroring PARs can be ignored since the sum over i will ultimately be applied. Therefore, the following can be read off the new mirroring PAR:

$$\{(1 + {}^gR_{1\ to\ t-1}) * ({}_iw_t * {}_iR_t) - (1 + {}^gR^B_{1\ to\ t-1}) * ({}_iw^B_t * {}_iR^B_t)\} =$$

$$= \{({}_iw_t - {}_iw^B_t) * [(1 + {}^gR^B_{1\ to\ t-1}) * {}_iR^B_t - (1 + {}^gR^B_{1\ to\ t-1}) * R^B_t]$$

$$+ {}_iw^B_t * [(1 + {}^gR_{1\ to\ t-1}) * {}_iR_t - (1 + {}^gR^B_{1\ to\ t-1}) * {}_iR^B_t]$$

$$+ ({}_iw_t - {}_iw^B_t) * [(1 + {}^gR_{1\ to\ t-1}) * {}_iR_t - (1 + {}^gR^B_{1\ to\ t-1}) * {}_iR^B_t]\}$$

$$+ \{({}_iw_t - {}_iw^B_t) * (1 + {}^gR^B_{1\ to\ t-1}) * R^B_t\}$$

$$= {}^{m}_{i}A_t + {}^{m}_{i}S_t + {}^{m}_{i}I_t + {}^{m}_{i}A0_t, \text{ and thus: } {}^gR_{1\ to\ T} - {}^gR^B_{1\ to\ T} = \Sigma_{t=1\ to\ T,\ i}\ \{{}^{m}_{i}A_t + {}^{m}_{i}S_t + {}^{m}_{i}I_t\}.$$

That is, the arithmetic excess of the geometric returns is the sum, over days and over industries, of the specified three rectangles of the new mirroring PAR (Figure 3 (*see page* 00)) representing the allocation, selection, and interaction contributions for industry i on day t.

APPENDIX 2

A Comparison of the Algebraic Structure of Various Approaches to Performance Attribution

2i

The arithmetic excess of arithmetic returns as a direct sum of the contributions (${}_iA_t$, ${}_iS_t$, and ${}_iI_t$) to the arithmetic excess of the incremental arithmetic returns (**AAAA** = Brinson):

$${}_iA_t = ({}_iw_t - {}_iw^B_t) * ({}_iR^B_t - R^B_t),$$

$${}_iS_t = {}_iw^B_t * ({}_iR_t - {}_iR^B_t),$$

$${}_iI_t = ({}_iw_t - {}_iw^B_t) * ({}_iR_t - {}_iR^B_t),$$

$${}_iC_t = {}_iA_t + {}_iS_t + {}_iI_t.$$

$$
\begin{aligned}
{}_iA_{1\text{ to }T} &= \Sigma_{t=1\text{ to }T}\ {}_iA_t, & A_t &= \Sigma_i\ {}_iA_t,\\
{}_iS_{1\text{ to }T} &= \Sigma_{t=1\text{ to }T}\ {}_iS_t, & S_t &= \Sigma_i\ {}_iS_t,\\
{}_iI_{1\text{ to }T} &= \Sigma_{t=1\text{ to }T}\ {}_iI_t. & I_t &= \Sigma_i\ {}_iI_t,\\
{}_iC_{1\text{ to }T} &= {}_iA_{1\text{ to }T} + {}_iS_{1\text{ to }T} + {}_iI_{1\text{ to }T}. & C_t &= A_t + S_t + I_t = {}^a\Delta^a{}_t = R_t - R^B{}_t.
\end{aligned}
$$

$$
\begin{aligned}
A_{1\text{ to }T} &= \Sigma_i\ {}_iA_{1\text{ to }T} &&= \Sigma_{t=1\text{ to }T}\ A_t,\\
S_{1\text{ to }T} &= \Sigma_i\ {}_iS_{1\text{ to }T} &&= \Sigma_{t=1\text{ to }T}\ S_t,\\
I_{1\text{ to }T} &= \Sigma_i\ {}_iI_{1\text{ to }T} &&= \Sigma_{t=1\text{ to }T}\ I_t,\\
C_{1\text{ to }T} &= A_{1\text{ to }T} + S_{1\text{ to }T} + I_{1\text{ to }T}.
\end{aligned}
$$

$$
\begin{aligned}
{}^aR_{1\text{ to }T} - {}^aR^B{}_{1\text{ to }T} &= \Sigma_{t=1\text{ to }T}\ \{R_t - R^B{}_t\} = \Sigma_{t=1\text{ to }T,\ i}\ \{{}_iw_t * {}_iR_t - {}_iw^B{}_t * {}_iR^B{}_t\}\\
&= \Sigma_{t=1\text{ to }T,\ i}\ \{{}_iC_t\} = \Sigma_i\ \{{}_iC_{1\text{ to }T}\} = \Sigma_{t=1\text{ to }T}\ \{C_t\} = C_{1\text{ to }T}.
\end{aligned}
$$

Since all the contributions (${}_iC_t$, ${}_iC_{1\text{ to }T}$, and C_t) are obtained, from ${}_iA_t$, ${}_iS_t$, and ${}_iI_t$, by sums, which all commute, the total excess return ($C_{1\text{ to }T} = {}^aR_{1\text{ to }T} - {}^aR^B{}_{1\text{ to }T}$) is produced by appropriately summing any set of contributions. This approach (**AAAA**), which is depicted by the PAR in Figure 1 (*see page* 00), analyzes the excess arithmetic (uncompounded) returns.

2ii

The geometric excess of geometric returns plus one as a product of a function of the contributions (${}_iA_t$, ${}_iS_t$, and ${}_iI_t$) to the arithmetic excess of the incremental arithmetic returns (**GGAA** = Multiplicative):

$$k_t = \{[(\ln(1 + R_t) - \ln(1 + R^B{}_t))/(R_t - R^B{}_t)].$$

$$
\begin{aligned}
{}^\pi{}_iA_t &= -1 + \exp[k_t * {}_iA_t],\\
{}^\pi{}_iS_t &= -1 + \exp[k_t * {}_iS_t],\\
{}^\pi{}_iI_t &= -1 + \exp[k_t * {}_iI_t\ ],\\
{}^\pi{}_iC_t &= -1 + (1 + {}^\pi{}_iA_t) * (1 + {}^\pi{}_iS_t) * (1 + {}^\pi{}_iI_t).
\end{aligned}
$$

$$
\begin{aligned}
{}^\pi{}_iA_{1\text{ to }T} &= -1 + \Pi_{t=1\text{ to }T}\ (1 + {}^\pi{}_iA_t), & {}^\pi A_t &= -1 + \Pi_i\ (1 + {}^\pi{}_iA_t),\\
{}^\pi{}_iS_{1\text{ to }T} &= -1 + \Pi_{t=1\text{ to }T}\ (1 + {}^\pi{}_iS_t), & {}^\pi S_t &= -1 + \Pi_i\ (1 + {}^\pi{}_iS_t),
\end{aligned}
$$

$^{\pi}{}_{i}I_{1\text{ to }T} = -1 + \Pi_{t=1\text{ to }T}\ (1 + {}^{\pi}{}_{i}I_{t})$, $^{\pi}I_{t} = -1 + \Pi_{i}\ (1 + {}^{\pi}{}_{i}I_{t})$,

$^{\pi}{}_{i}C_{1\text{ to }T} = -1 + (1 + {}^{\pi}{}_{i}A_{1\text{ to }T}) * (1 + {}^{\pi}{}_{i}S_{1\text{ to }T}) * (1 + {}^{\pi}{}_{i}I_{1\text{ to }T})$. $^{\pi}C_{t} = -1 + (1 + {}^{\pi}A_{t}) * (1 + {}^{\pi}S_{t}) * (1 + {}^{\pi}I_{t})$,

$= -1 + \exp[k_{t} * {}^{a}\Delta^{a}{}_{t}] = (1 + R_{t})/(1 + R^{B}{}_{t}) - 1$.

$^{\pi}A_{1\text{ to }T} = -1 + \Pi_{i}\ (1 + {}^{\pi}{}_{i}A_{1\text{ to }T}) = -1 + \Pi_{t=1\text{ to }T}\ (1 + {}^{\pi}A_{t})$,

$^{\pi}S_{1\text{ to }T} = -1 + \Pi_{i}\ (1 + {}^{\pi}{}_{i}S_{1\text{ to }T}) = -1 + \Pi_{t=1\text{ to }T}\ (1 + {}^{\pi}S_{t})$,

$^{\pi}I_{1\text{ to }T} = -1 + \Pi_{i}\ (1 + {}^{\pi}{}_{i}I_{1\text{ to }T}) = -1 + \Pi_{t=1\text{ to }T}\ (1 + {}^{\pi}I_{t})$,

$^{\pi}C_{1\text{ to }T} = -1 + (1 + {}^{\pi}A_{1\text{ to }T}) * (1 + {}^{\pi}S_{1\text{ to }T}) * (1 + {}^{\pi}I_{1\text{ to }T})$

$$(1 + {}^{g}R_{1\text{ to }T})/(1 + {}^{g}R^{B}{}_{1\text{ to }T}) = \Pi_{t=1\text{ to }T}\ \{(1 + R_{t})/(1 + R^{B}{}_{t})\} = \Pi_{t=1\text{ to }T}\ \{[1 + \Sigma_{i}({}_{i}w_{t} * {}_{i}R_{t})]/[1 + \Sigma_{i}({}_{i}w_{t} * {}_{i}R^{B}{}_{t})]\}$$

$$= \Pi_{t=1\text{ to }T,\, i}\ \{1 + {}^{\pi}{}_{i}C_{t}\} = \Pi_{i}\ \{1 + {}^{\pi}{}_{i}C_{1\text{ to }T}\} = \Pi_{t=1\text{ to }T}\ \{1 + {}^{\pi}C_{t}\} = 1 + {}^{\pi}C_{1\text{ to }T}$$

Since all the contributions plus one ($1 + {}^{\pi}{}_{i}C_{t}$, $1 + {}^{\pi}{}_{i}C_{1\text{ to }T}$, and $1 + {}^{\pi}C_{t}$) are obtained, from $1 + {}^{\pi}{}_{i}A_{t}$, $1 + {}^{\pi}{}_{i}S_{t}$, and $1 + {}^{\pi}{}_{i}I_{t}$, by taking their products, which all commute, the total excess return plus one, $[1 + {}^{\pi}C_{1\text{ to }T} = (1 + {}^{g}R_{1\text{ to }T})/(1 + {}^{g}R^{B}{}_{1\text{ to }T})]$, is produced upon appropriately multiplying any single set of one plus the contributions.

This approach to attribution gives rise to score keeping which is thoroughly multiplicative. Consequently, there is no PAR for this case. It also makes it comparatively more difficult to intuitively grasp the quantitative relationships between the contribution values it implies and the excess return the values are meant to explicate.

2iii

The geometric excess of geometric returns plus one as a product of one plus the contributions (${}^{BKT}A_{t}$, ${}^{BKT}S_{t}$, and ${}^{BKT}I_{t}$) to the geometric excess of the incremental arithmetic returns plus one, (**GGGA** = BKT):

$^{BKT}{}_{i}A_{t} = {}_{i}A_{t}/(1 + R^{B}{}_{t})$,

$^{BKT}{}_{i}S_{t} = {}_{i}S_{t}/(1 + R^{B}{}_{t})$,

$^{BKT}{}_{i}I_{t} = ?$,

$^{BKT}{}_{i}C_{t} = ?$.

$^{BKT}{}_{i}A_{1\text{ to }T} = ?$, $^{BKT}A_{t} = \Sigma_{i}\ {}^{BKT}{}_{i}A_{t}$,

$^{BKT}{}_{i}S_{1\text{ to }T} = ?$, $^{BKT}S_{t} = \Sigma_{i}\ {}^{BKT}{}_{i}S_{t}$,

$^{BKT}{}_{i}I_{1\text{ to }T} = ?$, $^{BKT}I_{t} = -1 + (1 + R_{t})/[(1 + R^{B}{}_{t}) * (1 + {}^{BKT}A_{t}) * (1 + {}^{BKT}S_{t})]$,

$^{BKT}{}_iC_{1\ to\ T}$ $= ?.$

$^{BKT}C_t = -1 + (1 + {}^{BKT}A_t) * (1 + {}^{BKT}S_t) * (1 + {}^{BKT}I_t)$

$= (1 + R_t)/(1 + R^B_t) - 1.$

$^{BKT}A_{1\ to\ T} = -1 + \Pi_{t = 1\ to\ T}\ (1 + {}^{BKT}A_t),$

$^{BKT}S_{1\ to\ T} = -1 + \Pi_{t = 1\ to\ T}\ (1 + {}^{BKT}S_t),$

$^{BKT}I_{1\ to\ T} = -1 + \Pi_{t = 1\ to\ T}\ (1 + {}^{BKT}I_t),$

$^{BKT}C_{1\ to\ T} = -1 + (1 + {}^{BKT}A_{1\ to\ T}) * (1 + {}^{BKT}S_{1\ to\ T}) * (1 + {}^{BKT}I_{1\ to\ T}).$

$(1 + {}^gR_{1\ to\ T})/(1 + {}^gR^B{}_{1\ to\ T}) = \Pi_{t = 1\ to\ T}\ \{(1 + R_t)/(1 + R^B_t)\} = \Pi_{t = 1\ to\ T}\ \{(1 + {}^{BKT}C_t)\} = 1 + {}^{BKT}C_{1\ to\ T}.$

It would be inconsistent with the above formulae to complete this chart by defining $^{BKT}{}_iA_{1\ to\ T}$ so that both:

$$^{BKT}{}_iA_{1\ to\ T} = -1 + \Pi_{t = 1\ to\ T}\ (1 + {}^{BKT}{}_iA_t) \text{ and } {}^{BKT}A_{1\ to\ T} = \Sigma_i\ {}^{BKT}{}_iA_{1\ to\ T}.$$

Similar problems exist for defining $^{BKT}{}_iS_{1\ to\ T}$, $^{BKT}{}_iI_{1\ to\ T}$ and $^{BKT}{}_iC_{1\ to\ T}$. Furthermore, there is also no natural way to define $^{BKT}{}_iI_t$ and $^{BKT}{}_iC_t$ so that they properly relate to $^{BKT}I_t$ and $^{BKT}C_t$, respectively.

The anti-intuitiveness of multiplicative score-keeping and the lack of coherent definitions of important properties severely militate against this approach. The lack of a corresponding PAR deprives it of an important explanatory aid.

An elegant variation on the BKT approach is offered by Carl Bacon (Bacon, 2000).

2iv

The geometric excess of geometric returns plus one as a product of one plus the contributions (^BA_t, and ^BSI_t) to the geometric excess of the incremental arithmetic returns plus one, (**GGGA2** = B):

${}_i^BA_t = {}_iA_t/(1 + R_t^B),$

${}_i^BSI_t = ({}_iS_t + {}_iI_t)/(1 + X_t),$

$X_t = \Sigma_i({}_iw_t^P * {}_iR_t^B).$

${}_i^BC_t = ?.$

${}_i^BA_{1\ to\ T} = ?,$

${}_i^BSI_{1\ to\ T} = ?,$

${}_i^BC_{1\ to\ T} = ?.$

$^BA_t = \Sigma_i\ ({}_i^BA_t),$

$^BSI_t = \Sigma_i\ ({}_i^BSI_t),$

$^BC_t = ({}^BA_t + 1) * ({}^BSI_t + 1) - 1 = (1 + R^P_t)/(1 + R^B_t) - 1.$

$^{B}A_{1\ to\ T} = -1 + \Pi_t (1 + {}^{B}A_t)$,

$^{B}SI_{1\ to\ T} = -1 + \Pi_t (1 + {}^{B}SI_t)$,

$^{B}C_{1\ to\ T} = ({}^{B}A_{1\ to\ T} + 1) * ({}^{B}SI_{1\ to\ T} + 1) - 1 = (1 + R^{P}_{1\ to\ T})/(1 + R^{B}_{1\ to\ T}) - 1$.

GGGA2 has characteristics very similar to that of **GGGA**.

2v

The arithmetic excess of geometric returns as a scaled sum of the contributions (${}_iA_t$, ${}_iS_t$, and ${}_iI_t$) to the arithmetic excess of the incremental arithmetic returns (**AGAA** = Carino/Russell):

$k_t/k = \{[\ln(1 + R_t) - \ln(1 + R^B_t)]/[R_t - R^B_t]\}/\{[\ln(1 + {}^{g}R_{1\ to\ T}) - \ln(1 + {}^{g}R^{B}_{1\ to\ T})]/[{}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T}]\}$.

${}^{R}_{i}A_t = k_t/k * {}_iA_t$,

${}^{R}_{i}S_t = k_t/k * {}_iS_t$,

${}^{R}_{i}I_t = k_t/k * {}_iI_t$,

${}^{R}_{i}C_t = {}^{R}_{i}A_t + {}^{R}_{i}S_t + {}^{R}_{i}I_t$.

${}^{R}_{i}A_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ {}^{R}_{i}A_t$,

${}^{R}_{i}S_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ {}^{R}_{i}S_t$,

${}^{R}_{i}I_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ {}^{R}_{i}I_t$,

${}^{R}_{i}C_{1\ to\ T} = {}^{R}_{i}A_{1\ to\ T} + {}^{R}_{i}S_{1\ to\ T} + {}^{R}_{i}I_{1\ to\ T}$.

${}^{R}A_t = \Sigma_i\ {}^{R}_{i}A_t$,

${}^{R}S_t = \Sigma_i\ {}^{R}_{i}S_t$,

${}^{R}I_t = \Sigma_i\ {}^{R}_{i}I_t$,

${}^{R}C_t = {}^{R}A_t + {}^{R}S_t + {}^{R}I_t$,

$= (k_t/k) * {}^{a}\Delta^{a}_t = (k_t/k) * (R_t - R^B_t)$.

${}^{R}A_{1\ to\ T} = \Sigma_i\ {}^{R}_{i}A_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ {}^{R}A_t$,

${}^{R}S_{1\ to\ T} = \Sigma_i\ {}^{R}_{i}S_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ {}^{R}S_t$,

${}^{R}I_{1\ to\ T} = \Sigma_i\ {}^{R}_{i}I_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ {}^{R}I_t$,

${}^{R}C_{1\ to\ T} = {}^{R}A_{1\ to\ T} + {}^{R}S_{1\ to\ T} + {}^{R}I_{1\ to\ T}$.

$${}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T} = \Sigma_{t=1\ to\ T}\ \{(k_t/k) * (R_t - R^B_t)\} = \Sigma_{t=1\ to\ T,\ i}\ \{(k_t/k) * ({}_iw_t * {}_iR_t - {}_iw^B_t * {}_iR^B_t)\}$$

$$= \Sigma_{t=1\ to\ T,\ i}\ \{{}^{R}_{i}C_t\} = \Sigma_i\ \{{}^{R}_{i}C_{1\ to\ T}\} = \Sigma_{t=1\ to\ T}\ \{{}^{R}C_t\} = {}^{R}C_{1\ to\ T}.$$

Since all the contributions (${}^{R}_{i}C_{t}$, ${}^{R}_{i}C_{1\ to\ T}$, and ${}^{R}C_{t}$) are obtained, from ${}^{R}_{i}A_{t}$, ${}^{R}_{i}S_{t}$ and ${}^{R}_{i}I_{t}$, by sums, which all commute, the total excess return (${}^{R}C_{1\ to\ T} = {}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T}$) is produced upon appropriately summing over any set of contributions. This approach is depicted by the PAR in Figure 2 (*see page* 00). The a-causality introduced through k is the most damaging drawback of this method.

Another variation on the BKT approach is sometimes employed.

2vi

The arithmetic excess of geometric returns as a sum of components (${}^{P}A_{1\ to\ T}$, ${}^{P}S_{1\ to\ T}$, and ${}^{P}Is_{1\ to\ T}$) which are functions of contributions to the arithmetic excess of the incremental arithmetic returns, (**AGAA2** = P):

$${}^{P}_{i}A_{t} = {}_{i}A_{t},$$

$${}^{P}_{i}S_{t} = {}_{i}S_{t},$$

$${}^{P}_{i}I_{t} = {}_{i}I_{t},$$

$${}^{P}_{i}C_{t} = {}_{i}A_{t} + {}_{i}S_{t} + {}_{i}I_{t}.$$

$${}^{P}_{i}A_{1\ to\ T} = -1 + \Pi_{t=1\ to\ T}\ (1 + {}^{P}_{i}A_{t}), \qquad {}^{P}A_{t} = ?,$$

$${}^{P}_{i}S_{1\ to\ T} = -1 + \Pi_{t=1\ to\ T}\ (1 + {}^{P}_{i}S_{t}), \qquad {}^{P}S_{t} = ?,$$

$${}^{P}_{i}I_{1\ to\ T} = -1 + \Pi_{t=1\ to\ T}\ (1 + {}^{P}_{i}I_{t}), \qquad {}^{P}I_{t} = ?,$$

$${}^{P}_{i}Is_{1\ to\ T} = {}^{P}_{i}I_{1\ to\ T}$$

$$+ [({}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T}) - ({}^{P}A_{1\ to\ T} + {}^{P}S_{1\ to\ T} + {}^{P}I_{1\ to\ T})]$$

$$* |{}^{P}_{i}I_{1\ to\ T}| / \Sigma_{i} |{}^{P}_{i}I_{1\ to\ T}|, \qquad {}^{P}Is_{t} = ?,$$

$${}^{P}_{i}C_{1\ to\ T} = {}^{P}_{i}A_{1\ to\ T} + {}^{P}_{i}S_{1\ to\ T} + {}^{P}_{i}Is_{1\ to\ T}. \qquad {}^{P}C_{t} = ?.$$

$${}^{P}A_{1\ to\ T} = \Sigma_{i}\ {}^{P}_{i}A_{1\ to\ T},$$

$${}^{P}S_{1\ to\ T} = \Sigma_{i}\ {}^{P}_{i}S_{1\ to\ T},$$

$${}^{P}I_{1\ to\ T} = \Sigma_{i}\ {}^{P}_{i}I_{1\ to\ T},$$

$${}^{P}Is_{1\ to\ T} = \Sigma_{i}\ {}^{P}_{i}Is_{1\ to\ T},$$

$${}^{g}R_{1\ to\ T} - {}^{g}R^{B}_{1\ to\ T} = {}^{P}A_{1\ to\ T} + {}^{P}S_{1\ to\ T} + {}^{P}Is_{1\ to\ T} = \Sigma_{i}\ \{{}^{P}_{i}C_{1\ to\ T}\} = {}^{P}C_{1\ to\ T}.$$

It would be inconsistent with the above formulae to complete this chart by defining ${}^{P}A_{t}$ so that both:

$${}^{P}A_t = \Sigma_i\, {}^{P}_{i}A_t \text{ and } {}^{P}A_{1\text{ to }T} = -1 + \Pi_{t=1\text{ to }T}\,(1 + {}^{P}A_t).$$

Similar problems exist for defining ${}^{P}S_t$, ${}^{P}I_t$, ${}^{P}Is_t$ and ${}^{P}C_t$. The anti-intuitiveness of multiplicative score-keeping and the lack of coherent definitions of important properties severely militate against this approach. The resulting lack of a corresponding PAR deprives it of an important explanatory aid.

2vii

The arithmetic excess of geometric returns as a direct sum of contributions (${}^{m}_{i}A_t$, ${}^{m}_{i}S_t$, and ${}^{m}_{i}I_t$) to the arithmetic excess of the incremental geometric returns (**AGAG** = mirrored):

$${}^{m}_{i}A_t = ({}_{i}w_t - {}_{i}w^B_t) * (1 + {}^{g}R^B_{1\text{ to }t-1}) * ({}_{i}R^B_t - R^B_t),$$

$${}^{m}_{i}S_t = {}_{i}w^B_t * [(1 + {}^{g}R_{1\text{ to }t-1}) * {}_{i}R_t - (1 + {}^{g}R^B_{1\text{ to }t-1}) * {}_{i}R^B_t],$$

$${}^{m}_{i}I_t = ({}_{i}w_t - {}_{i}w^B_t) * [(1 + {}^{g}R_{1\text{ to }t-1}) * {}_{i}R_t - (1 + {}^{g}R^B_{1\text{ to }t-1}) * {}_{i}R^B_t],$$

$${}^{m}_{i}C_t = {}^{m}_{i}A_t + {}^{m}_{i}S_t + {}^{m}_{i}I_t.$$

$${}^{m}_{i}A_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\, {}^{m}_{i}A_t,$$

$${}^{m}_{i}S_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\, {}^{m}_{i}S_t,$$

$${}^{m}_{i}I_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\, {}^{m}_{i}I_t,$$

$${}^{m}_{i}C_{1\text{ to }T} = {}^{m}_{i}A_{1\text{ to }T} + {}^{m}_{i}S_{1\text{ to }T} + {}^{m}_{i}I_{1\text{ to }T}.$$

$${}^{m}A_t = \Sigma_i\, {}^{m}_{i}A_t,$$

$${}^{m}S_t = \Sigma_i\, {}^{m}_{i}S_t,$$

$${}^{m}I_t = \Sigma_i\, {}^{m}_{i}I_t,$$

$${}^{m}C_t = {}^{m}A_t + {}^{m}S_t + {}^{m}I_t = {}^{a}\Delta^{g}_t.$$

$$= (1 + {}^{g}R_{1\text{ to }t-1}) * R_t - (1 + {}^{g}R^B_{1\text{ to }t-1}) * R^B_t.$$

$${}^{m}A_{1\text{ to }T} = \Sigma_i\, {}^{m}_{i}A_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\, {}^{m}A_t,$$

$${}^{m}S_{1\text{ to }T} = \Sigma_i\, {}^{m}_{i}S_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\, {}^{m}S_t,$$

$${}^{m}I_{1\text{ to }T} = \Sigma_i\, {}^{m}_{i}I_{1\text{ to }T} = \Sigma_{t=1\text{ to }T}\, {}^{m}I_t,$$

$${}^{m}C_{1\text{ to }T} = {}^{m}A_{1\text{ to }T} + {}^{m}S_{1\text{ to }T} + {}^{m}I_{1\text{ to }T}.$$

$$\begin{aligned}
{}^{g}R_{1\text{ to }T} - {}^{g}R^B_{1\text{ to }T} &= \Sigma_{t=1\text{ to }T}\, \{(1 + {}^{g}R_{1\text{ to }t-1}) * R_t - (1 + {}^{g}R^B_{1\text{ to }t-1}) * R^B_t\} \\
&= \Sigma_{t=1\text{ to }T,\, i}\, \{(1 + {}^{g}R_{1\text{ to }t-1}) * {}_{i}w_t * {}_{i}R_t - (1 + {}^{g}R^B_{1\text{ to }t-1}) * {}_{i}w_t * {}_{i}R^B_t\} \\
&= \Sigma_{t=1\text{ to }T,\, i}\, \{{}^{m}_{i}C_t\} = \Sigma_i\, \{{}^{m}_{i}C_{1\text{ to }T}\} = \Sigma_{t=1\text{ to }T}\, \{{}^{m}C_t\} = {}^{m}C_{1\text{ to }T}.
\end{aligned}$$

Since all the contributions (${}^{m}_{i}C_t$, ${}^{m}_{i}C_{1\text{ to }T}$, and ${}^{m}C_t$) are obtained, from ${}^{m}_{i}A_t$, ${}^{m}_{i}S_t$, and ${}^{m}_{i}I_t$, by sums, which all commute, the total excess return (${}^{m}C_{1\text{ to }T} = {}^{g}R_{1\text{ to }T} - {}^{g}R^B_{1\text{ to }T}$) is produced by appropriately summing over any single set of these contributions. This superior new mirroring approach is not a-causal and can also be depicted by a PAR, as shown in Figure 3 (*see page* 00).

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ENDNOTES

[1] Los (1999, pp. 169) interprets attribution methods that average over a single time period as "exact" by assuming that the time periods are infinitely small. He then employs a "first order approximation" in order to apply his method to the actual finite time periods required by the fact that portfolios are never continuously priced (Los 1999 Endnote 11). Thus, his "exact" method eventually leads to the residues or error terms,

which it is one of the aims of this paper to avoid.

[2] It is the standard to denote the endeavor to attribute (excess) return to variously defined components assessing managerial decisions by the name "performance attribution" (Brinson, 1991, pp. 47; Carino, 1999). Sometimes the term is also used to refer to the endeavor to attribute (excess) return to components assessing the predictive impact of historical properties of the market. This paper only addresses the first use of the term.

[3] "To permit an unequivocal evaluation of each type of fund management decision, an attribution analysis should avoid error terms or un-attributable, ambiguous components of management value," (Burnie, 1998, pp. 59).

[4] BKT (Burnie, 1998), and its relations **AGAA2** and **GGGA2** (cf. Appendix 2 (*see page* 00)), are exceptions. However, I will not consider them in the body of this paper since they are incomplete. BKT and **GGGA2** cannot meaningfully define, in a manner which combines to give the total excess return, the allocation or selection contribution of an individual industry for the whole period, nor can BKT meaningfully define the interaction contribution of an individual industry even for a single day. **AGAA2** cannot meaningfully define important components at the daily level. This will all be made explicit in Appendix 2 (*see page* 00).

[5] Carino (1999) states "… it is natural to assume that … 'Total' represents the sum," (pp. 6-7) and "We have found that plan sponsors and consultants prefer the familiar additive presentation … over other forms" (pp. 7). Surz (Surz 1999, pp. 14) states "Cumulative period analyses are complicated by the fact that returns are compounded, but it is desirable to have attribution components added."

[6] Dr. Jose G. Menchero (2000) has published a similar a-causal model, only his coefficients are intended to have a smaller range of variation than those (k_t/k) proposed by Carino.

[7] It is crucial to note that ${}^{g}R_{1\,to\,T}$, which is defined in terms of a product, is able to be written as a sum. An internal memo of Eric Fisher and Paul Davis (Fisher, 1985, pp. 7-8) first suggested the usefulness of this important identity to me.

[8] In addition to being mathematically permitted, omitting ${}_{i}A0_{t}$ is economically appropriate. If an industry in the benchmark earns the same return as the total benchmark it should not be assigned a positive contribution just because its portfolio weight is greater than its benchmark weight, or a negative contribution if its portfolio weight is less. Such terms will just cancel each other out, since both the portfolio and the benchmark weights must separately sum to one. Thus, the total benchmark return, and not zero return, is rightfully taken as the reference level to which benchmark industry returns are compared.

[9] One reason for this is that the information about how the weights evolve during the period is missing from the calculation of the period taken as a whole. Take the simple arithmetic approach (**AAAA**) as an example. For the individual two days, t = 1 and t = 2, the stock selection for industry i is: ${}_{i}S_{1} + {}_{i}S_{2} = {}_{i}w^{B}_{1} * ({}_{i}R_{1} - {}_{i}R^{B}_{1}) + {}_{i}w^{B}_{2} * ({}_{i}R_{2} - {}_{i}R^{B}_{2})$. However, for the two-day period taken as a whole the stock selection for industry i is: ${}_{i}S_{1+2} = {}_{i}w^{B}_{1+2} * ({}_{i}R_{1+2} - {}_{i}R^{B}_{1+2}) = {}_{i}w^{B}_{1} * [({}_{i}R_{1} + {}_{i}R_{2}) - ({}_{i}R^{B}_{1} + {}_{i}R^{B}_{2})] = {}_{i}w^{B}_{1} * ({}_{i}R_{1} - {}_{i}R^{B}_{1}) + {}_{i}w^{B}_{1} * ({}_{i}R_{2} - {}_{i}R^{B}_{2})$. Note that the industry weight, ${}_{i}w^{B}_{2}$, for the second day does not appear in the determination for the value of the stock selection for the two-day period taken as a whole. Only the weight, ${}_{i}w^{B}_{1+2} = {}_{i}w^{B}_{1}$, at the start of the period appears.

[10] The numerical results here exhibited for **AGAA** in Table 1 (*see page* 00) differ from those Continuously Compounding Effects presented by Carino (1999) in his Table 4 (*see page* 00). Carino's stated (1999, pp. 7) desirable goal is to obtain additive tables. A table of single-day results that add to give the "Additive Effect" for the total time period requires that the results he assigns to a single day are already divided by k. Carino's Table 4 exhibits Continuously Compounding Effects that are not yet divided by k, and, thus, do not represent the full additive contribution of the single day to the Additive Effect he displays in his Table 5 (*see page* 00) for the total time period. Here, Table 1, on the other hand, does present the full contribution of a single day to the total time period and, thus, the values of **AGAA** for a single day in Table 1 are simply the values of Carino's Continuously Compounding Effects of Table 4 divided by k.

[11] This might suggest some connection between dollar weighted returns and the identity: ${}^{g}R_{1\,to\,t} = \Sigma_{\tau=1\,to\,t}[(1+{}^{g}R_{1\,to\,\tau-1}) * R_{\tau}]$, which is the source of the presence of the geometric factor. However, any such connection is spurious, as can be seen by the fact that whether or not there are cash flows in or out of the portfolio the identity gives the time-weighted result.

[12] One can implement this further decomposition of the **AGAG** results, at all levels, by separating each individual contribution into the corresponding **AAAA** result plus the arithmetic difference between the two. The difference provides the contribution purely due to compounding. The **AAAA** result provides the non-compounded portion of the contribution, which is a measure of the contribution of all the days in isolation from each other and, for example, is never negative on a day if the corresponding contributions for that day is nonnegative. This further decomposition still retains all the advantages of **AGAG** listed in the conclusion while providing additional analysis of the decision structure.

$(1 + {}^{g}R^{B}_{1\ to\ t-1})*{}_{i}R^{B}_{t}$

${}^{m}_{i}A_{t} + {}^{m}_{i}S_{t} + {}^{m}_{i}I_{t}$

$(1 + {}^{g}R_{1\ to\ t-1})*{}_{i}R_{t}$

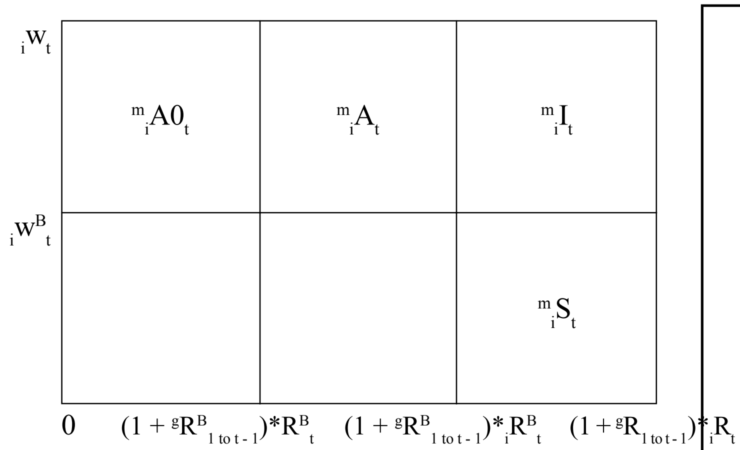

Figure 3
The New Mirroring PAR (AGAG).

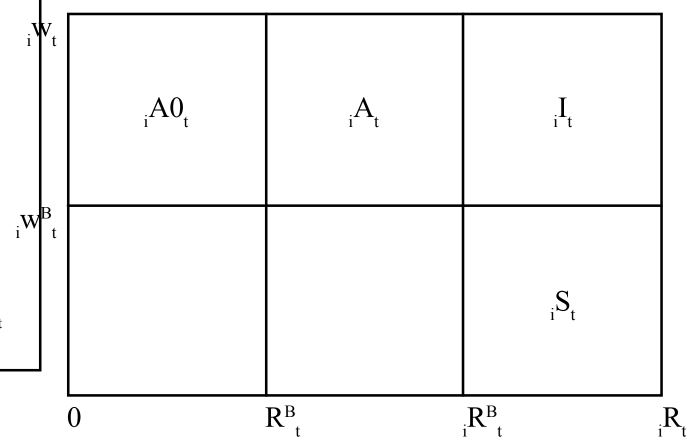

Figure 1
The Arithmetic PAR (AAAA).

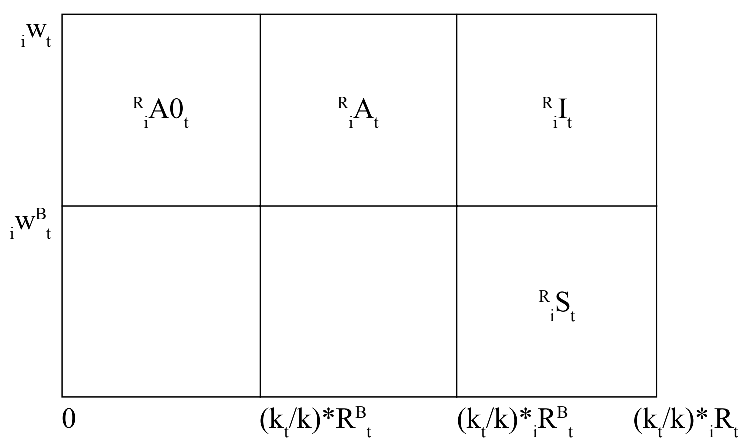

Figure 2
The Carino/Russell PAR (AGAA).

Table 3a
The Importance of the Basis for the Incremental Excesses.

	Portfolio Return for Day t.			Portfolio Market Value at Close of Day t.		Benchmark Return for Day t.			Benchmark Market Value at Close of Day t.
t	R	1+R	1+gR	MV	t	R^B	1+R^B	1+gR^B	MV^B
0		1.00	1.00	$100.00	0		1.0	1.00	$100.00
1	**15%**	1.15	1.15	$115.00	1	**10%**	1.1	1.10	$110.00
2	-60%	0.40	0.46	$46.00	2	10%	1.1	1.21	$121.00
3	**15%**	1.15	0.53	$52.90	3	**10%**	1.1	1.33	$133.10

t	Excess Arithmetic Return for Day t. R-R^B	Excess Geometric Return for Day t. gR-gR^B	Incremental Excess Geometric Return for Day t. D(gR-gR^B)	Excess Market Value at Close of Day t. MV - MV^B	Incremental Excess Market Value at Close of Day t. D(MV - MV^B)
0	0.00%	0.00%	0.00%	$0.00	$0.00
1	5.00%	5.00%	**5.00%**	$5.00	**$5.00**
2	-70.00%	-75.00%	-80.00%	-$75.00	-$80.00
3	5.00%	-80.20%	**-5.20%**	-$80.20	**-$5.20**
sum =	-60.00%		-80.20%		-$80.20
		(1+gR)/(1+gR^B) -1=	-60.26%		

For every $100 the portfolio started with it lost $80.20 relative to the benchmark over the total period. This loss came from a $5.00 relative gain on the first day; an $80.00 relative loss on the second day; and a $5.20 relative loss on the third day.

Table 3b
The Components Proposed by the Different Approaches for Explaining the Excess Incremental Returns

t	AAAA R-R^B	GGAA & GGGA (1+R)/(1+R^B)-1	AGAA (kt/k)*(R-R^B)	AGAG (1+gR(t-1))*R-(1+gR^B(t-1))*R^B
0				
1	5.00%	4.55%	3.86%	**5.00%**
2	-70.00%	-63.64%	-87.93%	-80.00%
3	5.00%	4.55%	3.86%	**-5.20%**
sum=	-60.00%		-80.20%	-80.20%
{Πt[(1+R)/(1+R^B)]} -1=		-60.26%		

While the components of both AGAA and AGAG match the change in the relative Market Value for the whole period, only AGAG's components match the daily change in relative Market Value and geometrically accumulated excess returns, as given in Appendix 5a.